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I.—ON THE THEORY OF LINEAR TRANSFORMATIONS.

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THE following investigations were suggested to me by a very elegant paper on the same subject, published in the *Journal* by Mr. Boole. The following remarkable theorem is there arrived at. If a rational homogeneous function U, of the n^{th} order, with the m variables $x, y \ldots$, be transformed by linear substitutions into a function V of the new variables, $\xi, \eta \ldots$; if, moreover, θU expresses the function of the coefficients of U, which, equated to zero, is the result of the elimination of the variables from the series of equations $d_x U = 0$, $d_y U = 0$, &c., and of course θV the analogous function of the coefficients of V: then $\theta V = E^{n\alpha} . \theta U$, where E is the determinant formed by the coefficients of the equations which connect $x, y \ldots$ with $\xi, \eta \ldots$, and $\alpha = (n-1)^{m-1}$.* In attempting to demonstrate this very beautiful property, it occurred to me that it might be generalised by considering for the function U, not a homogeneous function of the n^{th} order between m variables, but one of the same order, containing n sets of (m) variables, and the variables of each set entering linearly. The form which Mr. Boole's theorem thus assumes is $\theta V = E_1^{\alpha} . E_2^{\alpha} \ldots E_n^{\alpha} . \theta U$. This it was easy to demonstrate would be true, if θU satisfied a certain system of partial differential equations. I imagined at first that these would determine the function θU, (supposed, in analogy with Mr. Boole's function, to represent the result of the elimination of the variables from $d_{x_1} U = 0$, $d_{y_1} U = 0 \ldots$ $d_{x_2} U = 0$, &c.): this I afterwards found was not the case; and thus I was led to a class of functions, including as a

* The value of α was left undetermined, but Mr. Boole has since informed me, he was acquainted with it at the time his paper was written; and has given it in a subsequent paper.

particular case the function θU, all of them possessed of the same characteristic property. The system of partial differential equations were without difficulty replaced by a more fundamental system of equations, upon which, assumed as definitions, the theory appears to me naturally to depend; and it is this view of it which I intend partially to develope in the present paper.

I have already employed the notation

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \ldots \\ \alpha', & \beta', & \gamma', & \delta' \ldots \\ \alpha'', & \beta'', & \gamma'', & \delta'' \ldots \\ \vdots & & & \end{matrix} \right\| \quad \ldots\ldots\ldots (1)$$

(where the number of horizontal rows is less than that of vertical ones) to denote the series of determinants,

$$\left| \begin{matrix} \alpha, & \beta, & \gamma \ldots \\ \alpha', & \beta', & \gamma' \ldots \\ \alpha'', & \beta'', & \gamma'' \ldots \\ \vdots & & \end{matrix} \right| \quad \ldots\ldots\ldots\ldots (2),$$

which can be formed out of the above quantities by selecting any system of vertical rows; these different determinants not being connected together by the sign +, or in any other manner, but being looked upon as perfectly separate.

The fundamental theorem for the multiplication of determinants gives, applied to these, the formula

$$\left\| \begin{matrix} A, & B, & C, & D \ldots \\ A', & B', & C', & D' \ldots \\ A'', & B'', & C'', & D'' \ldots \\ \vdots & & & \end{matrix} \right\| = E \left\| \begin{matrix} \alpha, & \beta, & \gamma, & \delta \ldots \\ \alpha', & \beta', & \gamma', & \delta' \ldots \\ \alpha'', & \beta'', & \gamma'', & \delta'' \ldots \\ \vdots & & & \end{matrix} \right\| \ldots (3),$$

where

$$\left. \begin{aligned} A &= \lambda\alpha + \lambda'\alpha' + \lambda''\alpha'' + \ldots \\ B &= \lambda\beta + \lambda'\beta' + \lambda''\beta'' + \ldots \\ &\vdots \\ A' &= \mu\alpha + \mu'\alpha' + \mu''\alpha'' + \ldots \\ B' &= \mu\beta + \mu'\beta' + \mu''\beta'' + \ldots \\ &\vdots \\ &\&c. \end{aligned} \right\} \ldots\ldots (4),$$

$$E = \left| \begin{matrix} \lambda, & \mu \ldots \\ \lambda', & \mu' \ldots \\ \vdots & \end{matrix} \right| \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

And the meaning of the equation is, that the terms on the first side are equal, each to each, to the terms on the second side.

This preliminary theorem being explained, consider a set of arbitrary coefficients, represented by the general formula

$$rst \ldots \ldots\ldots\ldots\ldots \quad (6),$$

in which the number of symbolical letters $r, s \ldots$ is n, and where each of these is supposed to assume all integer values, from 1 to m inclusively.

Let

$$as_{,}'t_{,}' \ldots, \quad as_{,}''t_{,}'' \ldots, \quad \ldots \ldots\ldots\ldots \quad (7)$$

represent the whole series, taken in any order, in which the first symbolical letter is a. Similarly,

$$r_{,,}'at_{,,}' \ldots, \quad r_{,,}''at_{,,}'', \quad \ldots \ldots\ldots\ldots \quad (8),$$

the whole series of those in which the second symbolical letter is a, and so on.

Imagine a function u of the coefficients, which is simultaneously of the forms

$$u = H_p \left\| \begin{matrix} 1s_{,}'t_{,}' \ldots, & 1s_{,}''t_{,}'' \ldots, & \ldots \\ 2s_{,}'t_{,}' \ldots, & 2s_{,}''t_{,}'' \ldots, & \ldots \\ \vdots & & \end{matrix} \right\| \ldots\ldots \quad (\mathrm{A}),$$

$$u = H_p \left\| \begin{matrix} r_{,,}'1t_{,,}' \ldots, & r_{,,}''1t_{,,}'' \ldots, & \ldots \\ r_{,,}'2t_{,,}' \ldots, & r_{,,}''2t_{,,}'' \ldots, & \ldots \\ \vdots & & \end{matrix} \right\|$$

&c.; in which H_p denotes a rational homogeneous function of the order p. The function H is not necessarily supposed to be the same in the above equations, and in point of fact it will not in general be so. The number of equations is of course (n).

The function (u), whose properties we proceed to investigate, may conveniently be named a "Hyperdeterminant." Any function satisfying any of the equations (A), without satisfying all of them, will be an "Incomplete Hyperdeterminant." But, considering in the first place such as are complete—

Let $\dot{r}st \ldots$ be a new set of coefficients connected with the former ones by a system of equations of the form

$$\dot{r}st \ldots = \lambda_1{}^r 1st \ldots + \lambda_2{}^r 2st \ldots + \lambda_m{}^r mst \ldots \ldots . (9),$$

(where the r in $\lambda_1{}^r \ldots$ is not an exponent, but an affix).

Suppose $\dot{u}$ is the same function of these new coefficients

that u was of the former ones. Then consider the first of the equations (A) and the equation (3), and writing

$$L = \begin{vmatrix} \lambda_1^1, & \lambda_1^2 \dots \\ \lambda_2^1, & \lambda_2^2 \dots \\ \vdots & \end{vmatrix} \dots\dots\dots\dots (10),$$

we have immediately the equation

$$\dot{u} = L^p . u \dots\dots\dots\dots\dots (11).$$

Consider the new set of coefficients

$$\ddot{r}\dot{s}t\dots = \mu_1{}^s r1t\dots + \mu_2{}^s r2t\dots + \dots + \mu_m{}^s rmt\dots \dots (12),$$

and $\ddot{u}$ the analogous function of these; then, from the second of the equations (A) and the equation (3), and writing

$$M = \begin{vmatrix} \mu_1^1, & \mu_1^2 \dots \\ \mu_2^1, & \mu_2^2 \dots \\ \vdots & \end{vmatrix} \dots\dots\dots (13),$$

$$\ddot{u} = M^p . \dot{u} = L^p M^p u \dots\dots\dots\dots (14).$$

In like manner, considering the new coefficients $\dddot{rst}\dots$, where

$$\dddot{rst}\dots = \nu_1{}^t rs1\dots + \nu_2{}^t rs2\dots \dots + \nu_n{}^t rsm\dots \dots (15),$$

the new function $\dddot{u}$ and the quantity N, given by

$$N = \begin{vmatrix} \nu_1^1, & \nu_1^2 \dots \\ \nu_2^1, & \nu_2^2 \dots \\ \vdots & \end{vmatrix} \dots\dots\dots\dots (16),$$

we have, as before,

$$\dddot{u} = N^p \ddot{u} = L^p M^p N^p u \dots\dots\dots (17),$$

or

$$\dddot{u} = L^p M^p N^p u \dots\dots\dots\dots\dots (18);$$

whence, generally, denoting the last result by u',

$$u' = L^p M^p N^p \dots u \dots\dots\dots\dots (B), (1).$$

Consider now the function

$$\Sigma\Sigma\Sigma\dots (rst\dots x_r y_s z_t \dots) \dots\dots\dots (19),$$

where the Σ's refer successively to $r, s, t\dots$, and denote summations from 1 to m inclusively. If u be looked upon as a derivative from the above function, we may write

$$u = \Theta . \Sigma\Sigma\Sigma\dots (rst\dots x_r y_s z_t \dots) \dots\dots (20).$$

Assume

$$\left.\begin{aligned} x_r &= \lambda_r^1 \dot{x}_1 + \lambda_r^2 \dot{x}_2 \dots + \lambda_r^m \dot{x}_m \\ y_s &= \mu_s^1 \dot{y}_1 + \mu_s^2 \dot{y}_2 \dots + \mu_s^m \dot{y}_m \\ &\vdots \end{aligned}\right\} \dots (21).$$

It is easy to obtain

$$\Sigma\Sigma\Sigma\ldots(rst\ldots\, x_r y_s z_t\ldots) = \Sigma\Sigma\Sigma\ldots(\dot{r}st\ldots\, \dot{x}_r y_s z_t\ldots)\ldots$$

$$= \Sigma\Sigma\Sigma\ldots(\dot{r}\dot{s}\dot{t}\ldots\, \dot{x}_r \dot{y}_s \dot{z}_t\ldots)\ldots \;\ldots\ldots\; (22),$$

and the formula for (u) becomes

$$\Theta\, \Sigma\Sigma\Sigma\ldots(\dot{r}\dot{s}\dot{t}\ldots\, \dot{x}_r \dot{y}_s \dot{z}_t)\ldots = L^p M^p N^p \ldots\, \Theta\, \Sigma\Sigma\Sigma\ldots(rst\ldots\, x_r y_s z_t\ldots)$$
$$\ldots\ldots\; (\mathrm{B}),\ (2).$$

Proceeding to obtain the expression of the coefficients $\dot{r}\dot{s}\dot{t}\ldots$ in terms of the coefficients $rst\ldots$, we have

$$\dot{r}\dot{s}\dot{t}\ldots = \Sigma\Sigma\Sigma\ldots(\lambda_f^{\,r} \mu_g^{\,s} \nu_h^{\,t} \ldots fgh\ldots)\ldots\ldots (\mathrm{C}),$$

where the Σ's refer successively to $f, g, h\ldots$, denoting summations from 1 to m inclusively. Having this equation, it is perhaps as well to retain

$$u' = L^p M^p N^p \ldots u \ldots\ldots\ldots\ldots (\mathrm{B}),\ (1),$$

instead of (B, 2), that form being principally useful in showing the relation of the function (u) to the theory of the transformation of functions.

It may immediately be seen, that in the equations (B, C) we may, if we please, omit any number of the marks of variation (.), omitting at the same time the corresponding signs Σ, and the corresponding factors of the series $L, M, N\ldots$

Also, if u be such as only to satisfy some of the equations (A); then, if in the same formulæ we omit the corresponding marks (.), summatory signs, and terms of the series $L, M, N\ldots$, the resulting equations are still true.

From the formulæ (A) we may obtain the partial differential equations

$$\Sigma\Sigma\ldots\left(\alpha st\ldots \frac{d}{d\beta st\ldots}\right) u = 0, \text{ or } pu\ldots\ldots(\mathrm{D}),$$

$$\Sigma\Sigma\ldots\left(r\alpha t\ldots \frac{d}{dr\beta t\ldots}\right) u = 0, \text{ or } pu,$$

according as α is not equal or is equal to β;

and so on: the summatory signs referring in every case to those of the series $r, s, t\ldots$, which are left variable, and extending from 1 to m inclusively.

To demonstrate this, consider the general form of u, as given by the first of the equations (A). This is evidently composed of a series of terms, each of the form

$$cPQR \ldots (p \text{ factors}).$$

In which $P = \begin{vmatrix} 1s_{,}'t_{,}', & 1s_{,}''t_{,}'' \; \ldots\ldots \; (m \text{ terms}) \\ \vdots & \\ \alpha s_{,}'t_{,}', & \alpha s_{,}''t_{,}'' \\ \vdots & \\ \beta s_{,}'t_{,}', & \beta s_{,}''t_{,}'' \\ \vdots & \end{vmatrix}$

Q, R, &c. being of the same form,

$$\Sigma\Sigma\ldots\left(\alpha st\ldots\frac{d}{d\beta st\ldots}\right)u$$

$$= cQR\ldots\;\Sigma\Sigma\ldots\left(\alpha st\ldots\frac{d}{d\beta st\ldots}\right)P + \&\text{c.} + \&\text{c.},$$

and

$$\Sigma\Sigma\ldots\left(\alpha st\ldots\frac{d}{d\beta st\ldots}\right)P = \begin{vmatrix} 1s_{,}'t_{,}'\ldots 1, & s_{,}''t_{,}'', \ldots (m \text{ terms}) \\ \vdots & \\ \alpha s_{,}'t_{,}'\ldots, & \alpha s_{,}''t_{,}'', \ldots \\ \vdots & \\ \alpha s_{,}'t_{,}'\ldots, & \alpha s_{,}''t_{,}'', \ldots \\ \vdots & \end{vmatrix} = 0;$$

so that all the terms on the second side of the equation vanish. If, however, $\beta = \alpha$,

$$\Sigma\Sigma\ldots\left(\alpha st\ldots\frac{d}{d\beta st\ldots}\right)P = P;$$

whence, on the second side, we have

$$cQR\ldots P + cPR\ldots Q + \&\text{c.} = p\,.\,cPQR\ldots + \&\text{c.} + \&\text{c.}$$

$$= pU,$$

or the theorem in question is proved.

In the case of an incomplete hyperdeterminant, the corresponding systems of equations are of course to be omitted. In every case it is from these equations that the form of the function (u) is to be investigated; they entirely replace the system (A).

A very important case of the general theory is, when we suppose the coefficients $rst\ldots$ to have the property $r's't'\ldots = r''s''t''\ldots$, whenever $r's't'\ldots$ and $r''s''t''\ldots$ denote the same combination of letters; and also that the coefficients λ are equal to the coefficients $\mu, \nu\ldots$, each to each. In this case the coefficients $\dot{r}\dot{s}\dot{t}\ldots$ have likewise the same property, viz. that $\dot{r}''\dot{s}''\dot{t}''\ldots = \dot{r}'\dot{s}'\dot{t}'\ldots$, whenever $r's't'\ldots$ and $r''s''t''\ldots$ denote the same combination of letters.

The equations (B, 1), (B, 2), become in this case

$$u' = L^{np}u \;\ldots\ldots\ldots\ldots\ldots\; (\text{B}, 3),$$

$$\Theta\,\Sigma\Sigma\Sigma\ldots\left\{\frac{[n]^n}{[a]^\alpha.[\beta]^\beta\ldots}\,\dot{r}\dot{s}\dot{t}\ldots\,\dot{x}_r\dot{x}_s\dot{x}_t\ldots\right\}$$

$$= L^{np}.\,\Theta\left\{\frac{[n]^n}{[a]^\alpha.[\beta]^\beta\ldots}\,rst\ldots\,x_rx_sx_t\ldots\right\}\ \ldots\ldots\ (\mathrm{B},\ 4).$$

Where only different combinations of values are to be taken for $r, s, t \ldots$ and $\alpha, \beta \ldots$, express how often the same number occurs in the series. In the equation (C), μ, ν must be replaced by λ, the equations (D) are no longer satisfied, the equations (A) reduce themselves to a single one, (so that there can be no question here of incomplete hyperdeterminants): but this is no longer sufficient to determine the function sought after. For this reason, the particular case, treated separately, would be far more difficult than the general one; but the formulæ of the general case being first established, these apply immediately to the particular one.* The case in question may be defined as that of symmetrical hyperdeterminants, (a denomination already adopted for ordinary determinants). It would be easily seen what on the same principle would be meant by partially symmetrical hyperdeterminants.

I have not yet succeeded in obtaining the general expression of a hyperdeterminant; the only cases in which I can do so are the following: I. $p=1$, n even, (if n be odd, there only exist incomplete hyperdeterminants). II. $p=2$, $m=2$, n even. III. $p=3$, $m=2$, $n=4$.

I. The first case is, in fact, that of the functions considered at the termination of a paper in the *Cambridge Philosophical Transactions*, Vol. VIII. Part I.; though at that time I was quite unacquainted with the general theory.

Using the notation there employed, we have

$$u = \left\{\begin{matrix} \dagger \\ 11\ldots(n) \\ 22 \\ :\,: \\ mm \end{matrix}\right\},$$

a complete hyperdeterminant when n is even; and when n is odd the functions

$$\left\{\begin{matrix} \dagger \\ 11\ldots(n) \\ 22 \\ :\,: \\ mm \end{matrix}\right\},\qquad \left\{\begin{matrix} \dagger \\ 11\ldots(n) \\ 22 \\ :\,: \\ mm \end{matrix}\right\}$$

are each of them incomplete hyperdeterminants.

* See Note at the end of this paper.

(A) In the case of $n = 2$, the complete hyperdeterminant is simply the ordinary determinant

$$\begin{vmatrix} 11, & 12 \ldots 1m \\ 21, & 22 \ldots 2m \\ \vdots & \\ m1, & m2 \ldots mm \end{vmatrix}.$$

Stating the general conclusion as applied to this case, which is a very well known one,

"If the function $U = \Sigma\Sigma\,(rs\,.\,x_r y_s)$

be transformed into a similar function

$$\Sigma\Sigma\,(\ddot{r}\ddot{s}\,.\,\dot{x}_r \dot{y}_s),$$

by means of the substitutions

$$x_r = \lambda_r^{\,1}\dot{x}_1 + \lambda_r^{\,2}\dot{x}_2 \ldots + \lambda_r^{\,m}\,.\,\dot{x}_m,$$
$$y_s = \mu_s^{\,1}\dot{y}_1 + \mu_s^{\,2}\dot{y}_2 \ldots + \mu_s^{\,m}\,.\,\dot{y}_m;$$

then $$\begin{vmatrix} \ddot{1}\ddot{1} & \ddot{1}\ddot{2}\ldots \\ \ddot{2}\ddot{1} & \ddot{2}\ddot{2} \\ \vdots & \end{vmatrix} = \begin{vmatrix} \lambda_1^{\,1}, & \lambda_2^{\,1}\ldots \\ \lambda_1^{\,2}, & \lambda_2^{\,2}\ldots \\ \vdots & \end{vmatrix} \begin{vmatrix} \mu_1^{\,1}, & \mu_2^{\,1}\ldots \\ \mu_1^{\,2}, & \mu_2^{\,2}\ldots \\ \vdots & \end{vmatrix} \begin{vmatrix} 11, & 12\ldots \\ 21, & 22\ldots \\ \vdots & \end{vmatrix}$$

Also, by what has preceded,

$$\ddot{r}\ddot{s} = \Sigma\Sigma\,(\lambda_f^{\,r}\,.\,\mu_g^{\,s}\,.fg);$$

so that the theorem is easily seen to amount to the following one—"If the terms of a determinant of the m^{th} order be of the form $\Sigma_r\Sigma_s\,(rs\,.\,x_{r\,.\,\rho}\,y_{s\,.\,\sigma})$, r, s extending as before, from 1 to m inclusively, the determinant itself is the product of three determinants; the first formed with the coefficients rs, the second with the quantities x, and the third with the quantities y."

In a following number of the *Journal* I shall prove, and apply to the theories of Maxima and Minima and of Spherical Co-ordinates, (I may just mention having obtained, in an elegant form, the formulæ for transforming from one oblique set of co-ordinates to another oblique one) the more general theorem,

"If k be the order of the determinant formed as above, the determinant itself is a quadratic function, its coefficients being determinants formed with the coefficients rs, its variables being determinants formed respectively with the variables x and the variables y; and the number of variables in each set being the number of combinations of k things out of m, ($= 1$ if $k = m$, if $k > m$ the determinant vanishes)."

I shall give in the same paper the demonstration of a very beautiful theorem, rather relating, however, to determinants than to quadratic functions, proved by Hesse in a Memoir in *Crelle's Journal*, vol. xx., "De curvis et superficiebus secundi ordinis;" and from which he has deduced the most interesting geometrical results. Another Memoir, by the same author, *Crelle*, vol. XXVIII., "Ueber die Elimination der Variabeln aus drei algeb. Gleichungen vom zweiter Grade mit zwei Variabeln," though relating in point of fact rather to functions of the third order, contains some most important results. A few theorems on quadratic functions, belonging, however, to a different part of the subject, will be found in my paper already quoted in the *Cambridge Philosophical Transactions;* and likewise in a paper in the *Journal* on the Algebraical Geometry of (n) dimensions.

I shall, just before concluding this case, write down the particular formula corresponding to three variables, and for the symmetrical case. It is, as is well known, the theorem,

"If $U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy$

be transformed into

$$\mathfrak{A}\xi^2 + \mathfrak{B}\eta^2 + \mathfrak{C}\theta^2 + 2\mathfrak{F}\eta\theta + 2\mathfrak{G}\xi\theta + 2\mathfrak{H}\xi\eta$$

by means of

$$\begin{aligned} x &= \alpha\xi + \beta\eta + \gamma\theta, \\ y &= \alpha'\xi + \beta'\eta + \gamma'\theta, \\ z &= \alpha''\xi + \beta''\eta + \gamma''\theta. \end{aligned}$$

Then $(\mathfrak{A}\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{F}^2 - \mathfrak{B}\mathfrak{G}^2 - \mathfrak{C}\mathfrak{H}^2 + 2\mathfrak{F}\mathfrak{G}\mathfrak{H})$

$$(\alpha\beta'\gamma'' - \alpha\beta''\gamma' + \alpha'\beta''\gamma - \alpha'\beta\gamma'' + \alpha''\beta\gamma' - \alpha''\beta'\gamma)^2$$
$$(ABC - AF^2 - BG^2 - CH^2 + 2FGH).$$

(B) Let $n = 3$, and for greater simplicity $m = 2$; write

$$\begin{aligned} a &= 111, & e &= 112, \\ b &= 211, & f &= 212, \\ c &= 121, & g &= 122, \\ d &= 221, & h &= 222. \end{aligned}$$

so that

$$\begin{aligned} U = {} & ax_1y_1z_1 + bx_2y_1z_1 + cx_1y_2z_1 + dx_2y_2z_1 \\ & + ex_1y_1z_2 + fx_2y_1z_2 + gx_1y_2z_2 + hx_2y_2z_2. \end{aligned}$$

There is no complete hyperdeterminant (i.e. for $p = 1$), and the incomplete ones are

$$\begin{aligned} ah - bg - cf + de &= u_{,}, \text{ suppose}, \\ ah - de - bg + cf &= u_{,,}, \\ ah - cf - de + bg &= u_{,,,}. \end{aligned}$$

Thus, suppose the transforming equations are

$$\begin{aligned} x_1 &= \lambda_1^{\,1}\dot{x}_1 + \lambda_1^{\,2}\dot{x}_2, \\ x_2 &= \lambda_2^{\,1}\dot{x}_1 + \lambda_2^{\,2}\dot{x}_2; \\ y_1 &= \mu_1^{\,1}\dot{y}_1 + \mu_1^{\,2}\dot{y}_2, \\ y_2 &= \mu_2^{\,1}\dot{y}_1 + \mu_2^{\,2}\dot{y}_2; \\ z_1 &= \nu_1^{\,1}\dot{z}_1 + \nu_1^{\,2}\dot{z}_2, \\ z_2 &= \nu_2^{\,1}\dot{z}_1 + \nu_2^{\,2}\dot{z}_2. \end{aligned}$$

Then

$\dddot{u}_{,} = (\mu_1^{\,1}\mu_2^{\,2} - \mu_2^{\,1}\mu_1^{\,2})(\nu_1^{\,1}\nu_2^{\,2} - \nu_2^{\,1}\nu_1^{\,2})\,u_{,}$, where y, z are changed,

$\dddot{u}_{,,} = (\nu_1^{\,1}\nu_2^{\,2} - \nu_2^{\,1}\nu_1^{\,2})(\lambda_1^{\,1}\lambda_2^{\,2} - \lambda_2^{\,1}\lambda_1^{\,2})\,u_{,,}$, " z, x "

$\dddot{u}_{,,,} = (\lambda_1^{\,1}\lambda_2^{\,2} - \lambda_2^{\,1}\lambda_1^{\,2})(\mu_1^{\,1}\mu_2^{\,2} - \mu_2^{\,1}\mu_1^{\,2})\,u_{,,,}$, " x, y "

We might also have assumed

$$\begin{aligned} u_{,} &= ad - bc, &&\text{or } eh - gf, \\ u_{,,} &= af - be, &&\text{or } ch - dg, \\ u_{,,,} &= ag - ce, &&\text{or } bh - df. \end{aligned}$$

But these are ordinary determinants.

(C). $n = 4$, $m = 2$.

$$\begin{aligned} a &= 1111, & i &= 1112, \\ b &= 2111, & j &= 2112, \\ c &= 1211, & k &= 1212, \\ d &= 2211, & l &= 2212, \\ e &= 1121, & m &= 1122, \\ f &= 2121, & n &= 2122, \\ g &= 2211, & o &= 2212, \\ h &= 2221, & p &= 2222. \end{aligned}$$

$$\begin{aligned} U = {} & ax_1y_1z_1w_1 + bx_2y_2z_1w_1 + cx_1y_2z_1w_1 + dx_2y_2z_1w_1 \\ & + ex_1y_1z_2w_1 + fx_2y_1z_2w_1 + gx_2y_2z_1w_1 + hx_2y_2z_2w_1 \\ & + ix_1y_1z_1w_2 + jx_2y_1z_1w_2 + kx_1y_2z_1w_2 + lx_2y_2z_1w_2 \\ & + mx_1y_1z_2w_2 + nx_2y_1z_2w_2 + ox_2y_2z_1w_2 + px_2y_2z_2w_2, \end{aligned}$$

we have $\quad u = ap - bo - cn + dm - el + fk + gj - hi.$

So that, with the same sets of transforming equations as above, and the additional one,

$$\begin{aligned} w_1 &= \rho_1^{\,1}\dot{w}_1 + \rho_1^{\,2}\dot{w}_2, \\ w_n &= \rho_2^{\,1}\dot{w}_1 + \rho_2^{\,2}\dot{w}_2, \end{aligned}$$

we have

$$\ddddot{u} = (\lambda_1^{\,1}\lambda_2^{\,2} - \lambda_2^{\,1}\lambda_1^{\,2})(\mu_1^{\,1}\mu_2^{\,2} - \mu_2^{\,1}\mu_1^{\,2})(\nu_1^{\,1}\nu_2^{\,2} - \nu_2^{\,1}\nu_1^{\,2})(\rho_1^{\,1}\rho_2^{\,2} - \rho_2^{\,1}\rho_1^{\,2})\,.\,u.$$

This is important when viewed in reference to a result which will presently be obtained.

If we take the symmetrical case, we have

$$U = ax^4 + 4\beta x^3y + 6\gamma x^2y^2 + 4\delta xy^3 + \varepsilon y^4;$$

which is transformed into

$$U' = a'x'^4 + 4\beta' x'^3y' + 6\gamma'^2 x'^2y'^2 + 4\delta' x'y'^3 + \varepsilon' y'^4,$$

by means of
$$x = \lambda x' + \mu y',$$
$$y = \lambda' x' + \mu' y'.$$

Then, if
$$u = a\varepsilon - 4\beta\delta + 3\gamma^2,$$
$$u' = a'\varepsilon' - 4\beta'\delta' + 3\gamma'^2,$$
$$u' = (\lambda\mu_{,} - \lambda_{,}\mu)^4 . u.$$

II. Where $p = 2$, $m = 2$, n is odd.

The expression

$$u = \left| \begin{matrix} \left\{\begin{matrix} \overset{\dagger}{1}111 \ldots (n) \\ 1222 \end{matrix}\right\}, & \left\{\begin{matrix} \overset{\dagger}{1}111 \ldots (n) \\ 2222 \end{matrix}\right\}, \\ \left\{\begin{matrix} \overset{\dagger}{1}111 \ldots (n) \\ 2222 \end{matrix}\right\}, & \left\{\begin{matrix} \overset{\dagger}{2}111 \ldots (n) \\ 2222 \end{matrix}\right\}, \end{matrix} \right|$$

is a complete hyperdeterminant; and that over whichever row the mark (†) of nonpermutation is placed. The different expressions so obtained are not, however, all of them independent functions. Thus, in the following example, where $n = 3$, the three functions are absolutely identical.

(*A*). $n = 3$, notation as in I. (*B*).

$$u = a^2h^2 + b^2g^2 + c^2f^2 + d^2e^2$$
$$- 2ahbg - 2ahcf - 2ahde - 2bgcf - 2bgde - 2cfde$$
$$+ 4adfg + 4bech.$$

and then

$$\dddot{u} = (\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2)^2 (\mu_1{}^1\mu_2{}^2 - \mu_2{}^1\mu_1{}^2)^2 (\nu_1{}^1\nu_2{}^2 - \nu_2{}^1\nu_1{}^2)^2 u.$$

This is in many respects an interesting example. We see that the function (u) may be expressed in the three following forms:

$$u = (ah - bg - cf + de)^2 + 4(ad - bc)(fg - eh) \ldots (1),$$
$$u = (ah - bg - de + cf)^2 + 4(af - be)(dg - ch) \ldots (2),$$
$$u = (ah - cf - de + bg)^2 + 4(ag - ce)(df - bh) \ldots (3),$$

which are indeed the direct results of the general form above given, the sign (†) being placed in succession over the different columns: and the three forms, as just remarked, are in this case identical.

We see from the first of these that u is of the second or third, from the second that u is of the first or third, from

the third that u is of the first or second of the three following forms:

$$u = H_2 \left\| \begin{matrix} a, b, c, d \\ e, f, g, h \end{matrix} \right\|, \quad u = H_2 \left\| \begin{matrix} a, b, e, f \\ c, d, g, h \end{matrix} \right\|, \quad u = H_2 \left\| \begin{matrix} a, c, e, g \\ b, d, f, h \end{matrix} \right\|$$

which is as it should be.

The following is a singular property of u.

Let $\quad a' = \frac{1}{2}\,\frac{du}{da}, \quad b' = \frac{1}{2}\,\frac{du}{db}, \; \ldots \; h' = \frac{1}{2}\,\frac{du}{dh}.$

Then, u' being the same function of these new coefficients that u is of the former ones,

$$u' = u^3.$$

To prove this, write

$$p = ah - bg - cf + de, \quad q = (ad - bc), \quad r = eh - fg.$$

$$\begin{aligned} a_, &= ap - 2q\,.\,e, & e_, &= -\,2ra + pe, \\ b_, &= bp - 2q\,.\,f, & f_, &= -\,2rb + pf, \\ c_, &= cp - 2q\,.\,g, & g_, &= -\,2rc + pg, \\ d_, &= dp - 2q\,.\,h, & h_, &= -\,2rd + ph. \end{aligned}$$

We have, as a particular case of the general formula just obtained, $\quad u_, = (p^2 - 4qr)^2\, u = u^2\,.\,u = u^3.$

Also
$$\begin{aligned} a_, &= h', & e_, &= d', \\ b_, &= -\,g', & f_, &= -\,c', \\ c_, &= -\,f', & g_, &= -\,b', \\ d_, &= e', & h_, &= a'; \end{aligned}$$

whence $u_, = u'$, i.e. $u' = u^3$.

There is no difficulty in showing also, that if $a'', b'' \ldots h''$ are derived from $a', b' \ldots h'$, as these are from $a, b \ldots h$, then

$$a'' = u^2 a, \; b' = u^2 b, \ldots . h'' = u^2 h.$$

The particular case of this theorem, which corresponds to symmetrical values of the coefficients, is given by M. Eisenstein, *Crelle*, vol. XXVII., as a corollary to his researches on the cubic forms of numbers.

Considering this symmetrical case

$$U = ax^3 + 3\beta x^2 y + 3\gamma x y^2 + \delta y^3,$$
$$u = a^2\delta^2 - 6a\delta\beta\gamma - 3\beta^2\gamma^2 + 4\beta^3\delta + 4a^3\gamma.$$

So that if U be transformed into

$$U' = a'x'^3 + 3\beta' x'^2 y' + 3\gamma' x' y'^2 + \delta' y'^3,$$

by means of
$$x = \lambda x' + \mu y',$$
$$y = \lambda_1 x' + \mu_1 y'.$$

And $\quad u' = a'^2\delta'^2 - 6a'\delta'\beta'\gamma' - 3\beta'^2\gamma'^2 + 4\beta'^3\delta' + 4a'^3\gamma',$

$$u' = (\lambda\mu_1 - \lambda_1\mu)^6\,.\,u.$$

III. $p = 3, \quad m = 2, \quad n = 4.$

Notation as in I(C),

$$
\begin{aligned}
u = {} & A\,(\mathfrak{A} + 3\mathfrak{B} + 3\mathfrak{C} + 6\mathfrak{D} + 6\mathfrak{E}) \\
& - B\,(\mathfrak{C} + \mathfrak{D} - 3\mathfrak{E} - \mathfrak{F} + 2\mathfrak{G} + 3\mathfrak{H}),
\end{aligned}
$$

where A, B, are arbitrary constants, and $\mathfrak{A}$, $\mathfrak{B}$, &c.$\ldots\mathfrak{H}$, are functions of the coefficients, given as follows:—

$$
\mathfrak{A} = a^3p^3 - b^3o^3 - c^3n^3 + d^3m^3 - e^3l^3 + f^3k^3 + g^3j^3 - h^3i^3.
$$

$$
\begin{aligned}
\mathfrak{B} = & - a^2p^2bo + b^2o^2ap + c^2n^2dm - d^2m^2cn + e^2l^2fk - elf^2k^2 - g^2j^2hi + gjh^2i^2 \\
& - a^2p^2cn + b^2o^2dm + c^2n^2ap - d^2m^2bo + e^2l^2gj - f^2k^2hi - g^2j^2el + h^2i^2fk \\
& - a^2p^2el + b^2o^2fk + c^2n^2gj - d^2m^2hi + e^2l^2ap - f^2k^2bo - g^2j^2cn + h^2i^2dm \\
& - a^2p^2hi + b^2o^2gj + c^2n^2fk - d^2m^2el + e^2l^2dm - f^2k^2cn - g^2j^2bo + h^2i^2ap.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{C} = & + a^2p^2dm - b^2o^2cn - c^2n^2bo + d^2m^2ap - e^2l^2hi + f^2k^2gj + fkg^2j^2 - h^2i^2el \\
& + a^2p^2fk - b^2o^2el - c^2n^2hi + d^2m^2gj - e^2l^2bo + f^2k^2ap + g^2j^2dm - h^2i^2cn \\
& + a^2p^2gj - b^2o^2hi - c^2n^2el + d^2m^2fk - e^2l^2cn + f^2k^2dm + g^2j^2ap - h^2i^2bo.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{D} = & \; apbocn - apbodm - apcndm + bocndm - elfkgj + elfkhi + elgjhi - higjfk \\
& + apboel - apbofk - cndmgj + dmcnhi - elfkap + elfkbo + gjhicn - higjdm \\
& - apbogj + apbohi + cndmel - dmcnkf + elfkcn - elfkdm - gjhiap + higjbo \\
& + apcnel - bodmfk - cnapgj + dmbohi - elgjap + fkhibo + gjelcn - hifkdm \\
& - apcnfk + bodmel + cnaphi - dmboel + elgjbo - fkhiap - gjeldm + hifkcn \\
& - apdmel + bocnkf + cmbogj - dmaphi + elhiap - fkgjbo - gjfkcn + hidmel.
\end{aligned}
$$

$$
\mathfrak{E} = apdmfk - bocnel - bocnih + dmapgj - elhibo + fkapgj + gjfkdm - hielcn.
$$

$$
\begin{aligned}
\mathfrak{F} = & \; a^2phjo - b^2goip - c^2nflm + d^2emkn - e^2dlkn + f^2ckml + g^2bjpi - h^2aijo \\
& - i^2phbg + j^2ogah + k^2nfde - l^2mecf + m^2ldcf - n^2ckde - o^2bjah + p^2aibg \\
& + a^2phkn - b^2golm - c^2nfip + d^2emjo - e^2dljo + f^2ckip + g^2bjpl - h^2aink \\
& - i^2phcf + j^2ogde + k^2nfah - l^2mebg + m^2ldbg - n^2ckah - o^2bjde + p^2aicf \\
& + a^2phlm - b^2gokn - c^2nfjo + d^2emip - e^2ldpi + f^2ckjo + g^2bjnk - h^2aiml \\
& - i^2phde + j^2gocf + k^2nfbg - l^2meah + m^2dlah - n^2ckbg - o^2bjcf + p^2aide \\
& + a^2pdno - b^2cmpo - c^2bpmn + d^2aomn - e^2hjkl + f^2gilk + g^2flij - h^2ekji \\
& - i^2lfgh + j^2kehg + k^2jhef - l^2igfe + m^2pbcd - h^2aodc - o^2ndab + p^2mbca \\
& + a^2plng - b^2hkmo - c^2ejpn + d^2fiom - e^2cpjl + f^2doik + g^2anlj - h^2bmki \\
& - i^2odfh + j^2pceg + k^2mhbf - l^2nage + m^2khbd - n^2lgac - o^2ifbd + p^2jeca \\
& + a^2plfo - b^2ekpo - c^2hjmn + d^2gimn - e^2bpkl + f^2aolk + g^2dnij - h^2cmji \\
& - i^2ndgh + j^2mchg + k^2pbef - l^2oafe + m^2jhcd - n^2igcd - o^2lfab + p^2keba.
\end{aligned}
$$

$$\begin{aligned}\mathfrak{G} = {} & apbgkn-boalhm-cndiep+dmcjfo-elfocj+fkpied+gjhmla-hignbk\\ & -ihjocf+gjidep+kflamh-lekbng+mdnbkg-ncmhal-obpied+paofjc\\ & +apbglm-boahkn-cndejo+dmcfip-elcfip+fkedoj+gjhakn-hibgml\\ & -ihjode+gjipcf+kflmbg-leknah+mdknah-ncmlbg-obpicf+paojde\\ & +apcflg-bojehk-cnjehk+dmilgf-elmpbc+fkadno+gjadno-hipmcb\\ & -ihadno+gjbmcp+kfcbpm-eladno+mdjehk-ncilfg-obilfg+pahejk\\ & +apcflm-bodkne-cnahjo+dmbgip-elbgip+fkahjo+gjednk-hicfml\\ & -ihknde+mdahjo+kfipbg-bocfml+gjcfml-ncpigb-leahjo+paknde\\ & +apidng-bojcmh-cnbkpe+dmaflo-elmhjc-fkidng+gjaflo-ihbkpe\\ & -ihaflo+gjbkpe+fkcjhm-elidng+mdbkpe-cnaflo-boidng+pahmcj\\ & +apidfo-bojcep-cnbkhm+dmalng-elmnbk-fkalng+gjidfo-ihpecj\\ & -ihalng+gjkbmh+fkcjpe-elidfo+mdjcep-ncidfo-obalng+pahmbk.\end{aligned}$$

$$\begin{aligned}\mathfrak{H} = {} & a^2honl-b^2pgmk-c^2pfmj+d^2onie-e^2dpkj+f^2ilco+g^2blni-h^2amkj\\ & -i^2pdfg+j^2oech+k^2nbeh-l^2mafg+m^2blch-n^2kadg-o^2jadf+p^2ibec:\end{aligned}$$

we have, as usual,

$$\ddddot{u} = (\lambda_1^{\,1}\lambda_2^{\,2}-\lambda_2^{\,1}\lambda_1^{\,2})^3.(\mu_1^{\,1}\mu_2^{\,2}-\mu_2^{\,1}\mu_1^{\,2})^3(\nu_1^{\,1}\nu_2^{\,2}-\nu_2^{\,1}\nu_1^{\,2})^3(\rho_1^{\,1}\rho_2^{\,2}-\rho_1^{\,2}\rho_2^{\,1})^3.u.$$

Particular forms of U are

$A = 1,\ B = 0$

$$\begin{aligned} u &= \mathfrak{A} + 3\mathfrak{B} + 3\mathfrak{C} + 6\mathfrak{D} + 6\mathfrak{E}\\ &= (ap-bo-cn+dm-el+fk+gj-hi)^3. = \nu^3.\ \text{suppose}\end{aligned}$$

$A = 1,\ B = 9.$

$$\begin{aligned} u &= \mathfrak{A} + 3\mathfrak{B} - 6\mathfrak{C} - 3\mathfrak{D} + 33\mathfrak{E} + 9\mathfrak{F} - 18\mathfrak{G} - 27\mathfrak{H}.\\ &= \theta U \ \text{suppose,}\end{aligned}$$

where $\theta U = 0$ is the result of the elimination of the variables from the equations $d_{x_1}U = 0$, $d_{y_1}U = 0$, $d_{z_1}U = 0$, $d_{w_1}U = 0$, $d_{x_2}U = 0$, $d_{y_2}U = 0$, $d_{z_2}U = 0$, $d_{w_2}U = 0$. In fact, by an investigation similar to Mr. Boole's, applied to a function such as U, it is shown that θU has the characteristic property of the function u: also in the present case (u) is the most general function of its kind, so that θU is obtained from U by properly determining the constant. This has been effected by comparing the value of u, in the symmetrical case, with the value of θU, in the same case, the expanded expression of which is given by Mr. Boole in the *Journal*, vol. IV. p. 169. Assuming $A = 1$, the result was $B = 9$.

The general form of u now becomes

$$u = a\nu^3 + \beta\theta U,$$

in which a, β, are indeterminate.

We have $\qquad \ddddot{u} = a\ddddot{\nu}^3 + \beta\theta\ddddot{U} = M(a\nu^3 + \beta\theta U).$

$$M = (\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2)^3 . (\mu_1{}^1\mu_2{}^2 - \mu_2{}^1\mu_1{}^2)^3 (\nu_1{}^1\nu_2{}^2 - \nu_2{}^1\nu_1{}^2)^3 (\rho_1{}^1\rho_2{}^2 - \rho_2{}^1\rho_1{}^2)^3.$$

And thence $\qquad \ddddot{\nu}^3 = M\nu^3,$

which coincides with a previous formula, and

$$\theta\ddddot{U} = M\theta U:$$

whence, eliminating M,

$$\frac{\theta\ddddot{U}}{\ddddot{\nu}^3} = \frac{\theta U}{\nu^3},$$

an equation which is remarkable as containing only the constants of U and $\ddddot{U}$: it is an equation of condition which must exist among the constants of $\ddddot{U}$ in order that this function may be derivable by linear substitutions from U.

In the symmetrical case, or where

$$U = ax^4 + 4\beta x^3y + 6\gamma x^2y^2 + 4\delta xy^3 + \varepsilon y^4.$$

It has been already seen that ν is given by

$$\nu = a\varepsilon - 4\beta\delta + 3\gamma^2.$$

Proceeding to form θU, we have

$\mathfrak{A} = a^3\varepsilon^3 - 4\beta^3\delta^3 + 3\gamma^6.$

$\mathfrak{B} = 4(a\varepsilon\beta^2\delta^2 - a^2\varepsilon^2\beta\delta + 3\gamma^2\beta^2\delta^2 - 3\beta\delta\gamma^4).$

$\mathfrak{C} = 3(a^2\varepsilon^2\gamma^2 + 2\gamma^6 + a\varepsilon\gamma^4 - 4\beta^3\delta^3).$

$\mathfrak{D} = 6(a\varepsilon\beta^2\delta^2 - 2a\varepsilon\beta\delta\gamma^2 + 3\beta^2\gamma^2\delta^2 - 2\beta\delta\gamma^4).$

$\mathfrak{E} = 3a\varepsilon\gamma^4 - 4\beta^3\delta^3 + \gamma^4.$

$\mathfrak{F} = 6(a^2\delta^2\gamma\varepsilon + \varepsilon^2\beta^2\gamma a - 2\beta^3\varepsilon\gamma\delta - 2\delta^3 a\beta\gamma - 4\beta^2\gamma^2\delta^2.$
$\qquad + 4.\beta\delta\gamma^4 + \gamma^3\beta^2\varepsilon + \gamma^3 a\delta^2).$

$\mathfrak{G} = 12(a\varepsilon\beta\delta\gamma^2 - a\beta\gamma\delta^3 - \varepsilon\gamma\delta\beta^3 + \gamma^3 a\delta^2 + \gamma^3\varepsilon\beta^2 + \beta\delta\gamma^4 - 2\beta^2\gamma^2\delta^2).$

$\mathfrak{H} = (a^2\delta^4 + \varepsilon^2\beta^4 - 4\beta^2\varepsilon\gamma^3 - 4a\gamma^3\delta^2 + 6\beta^2\gamma^2\delta^2).$

And these values give

$$\begin{aligned}\theta U = a^3\varepsilon^3 &- 6a\beta^2\delta^2\varepsilon - 12a^2\beta\delta\varepsilon^2 - 18a^2\gamma^2\varepsilon^2 - 27a^2\delta^4 - 27\beta^4\varepsilon^2.\\ &+ 36\beta^2\gamma^2\delta^2 + 54.a^2\gamma\delta^2\varepsilon + 54.a\beta^2\gamma\varepsilon^2 - 54a\gamma^3\delta^2 - 54\beta^2\gamma^3\varepsilon\\ &- 64\beta^3\delta^3 + 81a\gamma^4\varepsilon + 108a\beta\gamma\delta^3 + 108\beta^3\gamma\delta\varepsilon - 180a\beta\gamma^2\delta\varepsilon.\end{aligned}$$

So that this function, divided by $(\alpha\epsilon - 4\beta\delta + 3\gamma^2)^3$, is invariable for all functions of the fourth order which can be deduced one from the other by linear substitutions. The function $\alpha\epsilon - 4\beta\delta + 3\gamma^2$ occurs in other investigations: I have met with it in a problem relating to a homogeneous function of two variables, of any order whatever, $\alpha, \beta, \gamma, \delta, \epsilon$ signifying the fourth differential coefficients of the function. But this is only remotely connected with the present subject.

Since writing the above, Mr. Boole has pointed out to me that in the transformation of a function of the fourth order of the form $ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4$—besides his function θu, and my quadratic function $ae - 4bd + 3c^2$,—there exists a function of the third order $ace - b^2e - ad^2 - c^3 + 2bdc$, possessing precisely the same characteristic property, and that, moreover, the function θu may be reduced to the form

$$(ae - 4bd + 3c^2)^3 - 27.(ace - ad^2 - eb^2 - c^3 + 2bdc)^2;$$

the latter part of which was verified by trial; the former he has demonstrated in a manner which, though very elegant, does not appear to be the most direct which the theorem admits of. In fact, it may be obtained by a method just hinted at by Mr. Boole, in his earliest paper on the subject, *Mathematical Journal*, vol. II. p. 70. The equations $d_x{}^2u = 0$, $d_xd_yu = 0$, $d_y{}^2u = 0$, imply the corresponding equations for the transformed function: from these equations we might obtain two relations between the coefficients, which, in the case of a function of the fourth order, are of the orders 3 and 4 respectively: these imply the corresponding relations between the coefficients of the transformed function. Let $A = 0$, $B = 0$, $A' = 0$, $B' = 0$, represent these equations; then, since $A = 0$, $B = 0$, imply $A' = 0$, we must have $A' = \Lambda A' + MB$, Λ, M, being functions of λ, λ', μ, &c. μ': but B being of the fourth order, while A, A' are only of the third order in the coefficients of u, it is evident that the term MB must disappear, or that the equation is of the form $A' = \Lambda A$. The function A is obviously the function which, equated to zero, would be the result of the elimination of x^2, xy, y^2, considered as independent quantities from the equations $ax^2 + 2bxy + cy^2 = 0$, $bx^2 + 2cxy + dy^2 = 0$, $cx^2 + 2dxy + ey^2 = 0$, viz. the function given above. Hence the two functions on which the linear transformation of functions of the fourth order ultimately depend are the very simple ones

$$ae - 4bd + 3c^2,\ ace - ad^2 - eb^2 - c^3 + 2bdc,$$

the function of the sixth order being merely a derivative from these. The above method may easily be extended: thus, for instance, in the transformation of functions of any even order, I am in possession of several of the transforming functions; that of the fourth order, for functions of the sixth order, I have actually expanded: but it does not appear to contain the complete theory. Again, in the particular case of homogeneous functions of two variables, the transforming functions may be expressed as symmetrical functions of the roots of the equations $u = 0$, which gives rise to an entirely distinct theory. This, however, I have not as yet developed sufficiently for publication. There does not appear to be anything very directly analogous to the subject of this note, in my general theory: if this be so, it proves the absolute necessity of a distinct investigation for the present case, the one which I have denominated the symmetrical one.

II.—ON MAGIC SQUARES.

By R. Moon, M.A. Fellow of Queens' College.

The theory of Magic Squares has long exercised the ingenuity of mathematicians. It will be my object in the present paper, rather to unfold a simplification of the theory itself than to present any new or striking additions to it.

The ordinal numbers, from 1 to 25 inclusive, may be represented by the formula

$$1 + x + 5y,$$

where x and y are independent, and may respectively take any of the values 0. 1. 2. 3. 4. That such will be the case may be easily seen by arranging the numbers in the following manner:

y					
0	1	2	3	4	5
1	6	7	8	9	10
2	11	12	13	14	15
3	16	17	18	19	20
4	21	22	23	24	25
	0	1	2	3	4 x

By means of the above formula we shall proceed to construct a magic square.

A	B	C
$1+x_0+5y_0$	$1+x_2+5y_3$	$1+x_4+5y_1$
$1+x_1+5y_1$	$1+x_3+5y_4$	$1+x_0+5y_2$
$1+x_2+5y_2$	$1+x_4+5y_0$	$1+x_1+5y_3$
$1+x_3+5y_3$	$1+x_0+5y_1$	$1+x_2+5y_4$
$1+x_4+5y_4$	$1+x_1+5y_2$	$1+x_3+5y_0$

D	E
$1+x_1+5y_4$	$1+x_3+5y_2$
$1+x_2+5y_0$	$1+x_4+5y_3$
$1+x_3+5y_1$	$1+x_0+5y_4$
$1+x_4+5y_2$	$1+x_1+5y_0$
$1+x_0+5y_3$	$1+x_2+5y_1$

The subscribed figures in the above columns denote the values to be assigned to x and y respectively. If the substitutions thus indicated be made, it will be found that the numbers contained in the columns *A, B, C, D, E,* (the columns being arranged side by side in the order of the letters), will constitute a magic square. This will be seen if we consider that,

(1) The sum of the numbers in each column

$$= 5 + (x_0 + x_1 + x_2 + x_3 + x_4) + 5\,(y_0 + y_1 + y_2 + y_3 + y_4).$$

(2) If we take the first number in each column and add them together, their sum will be equal to the sum of each column taken vertically; and so of the sums of the second, third, &c. numbers respectively.

(3) The sum of the first number of *A*, the second of *B*, the third of *C*, the fourth of *D*, and the fifth of *E*, will likewise be equal to the same quantity; as also will be the sum of the fifth of *A*, the fourth of *B*, the third of *C*, the second of *D*, and the first of *E*.

A slight inspection of the columns will shew how they may be successively derived one from another. It will also be seen that by the same rule that *B* is derived from *A*, *C* from *B*, and so on, we may also derive *A* from *E*; so that the arrangement is (if we may so term it) circular: and of course we can go backwards in the circle as well as forwards, *i.e.* we may derive *B* from *C* as easily as we can derive *C* from *B*.

It will also be seen that, the order of the columns being preserved, it is quite indifferent which we place at the side or which we begin with. It is further to be observed that the sole limitation to be attended to in the formation of the first or *generating* column is, that its sum must be

$$= 5 + (x_0 + x_1 + x_2 + x_3 + x_4) + 5\,(y_0 + y_1 + y_2 + y_3 + y_4).$$

The order in which the values of x and y are assigned is indifferent. It is only requisite that no two numbers of the generating column contain the same value of x or the same value of y.

The number of magic squares which may be formed according to the above method is equal to the number of ways in which the *generating* column may be formed, *i.e.* $= (5.4.3.2.1)^2$; or, if we consider that each square will substantially recur four times, the only difference being according as we take the numbers from right to left or from top to bottom, the number of different squares will be $= \left(\dfrac{5.4.3.2.1}{2}\right)^2$.

The following arrangement will likewise constitute a magic square:

A	*B*	*C*
$1 + x_3 + 5y_2$	$1 + x_4 + 5y_1$	$1 + x_0 + 5y_0$
$1 + x_4 + 5y_3$	$1 + x_0 + 5y_2$	$1 + x_1 + 5y_1$
$1 + x_0 + 5y_4$	$1 + x_1 + 5y_3$	$1 + x_2 + 5y_2$
$1 + x_1 + 5y_0$	$1 + x_2 + 5y_4$	$1 + x_3 + 5y_3$
$1 + x_2 + 5y_1$	$1 + x_3 + 5y_0$	$1 + x_4 + 5y_4$

D	*E*
$1 + x_1 + 5y_4$	$1 + x_2 + 5y_3$
$1 + x_2 + 5y_0$	$1 + x_3 + 5y_4$
$1 + x_3 + 5y_1$	$1 + x_4 + 5y_0$
$1 + x_4 + 5y_2$	$1 + x_0 + 5y_1$
$1 + x_5 + 5y_3$	$1 + x_1 + 5y_2$

The mode of successive formation in this case is obvious. The arrangement here, as in the former example, is circular; but we are not at liberty in this case, as in the last, to begin with any column in the series. *C* must necessarily be the middle column. The reason of this will readily appear. For if we take *D* for the middle column, the sum of the places in the diagonals of the square (which will be found by taking the sum of the first number of *B*, the second of *C*, the third of *D*, the fourth of *E*, and the fifth of *A*, and the sum of the

fifth of *B*, the fourth of *C*, the third of *D*, the second of *E*, and the first of *A*,) will be respectively

$$5 + (x_4 + x_1 + x_3 + x_0 + x_2) + 5\,(y_1 + y_1 + y_1 + y_1 + y_1),$$

$$5 + (x_3 + x_3 + x_3 + x_3 + x_3) + 5\,(y_0 + y_3 + y_1 + y_4 + y_2).$$

Hence this arrangement fails. But if *C* be the middle column, the sums of the diagonals are, respectively,

$$5 + (x_3 + x_0 + x_2 + x_4 + x_1) + 5\,(y_2 + y_2 + y_2 + y_2 + y_2),$$

$$5 + (x_2 + x_2 + x_2 + x_2 + x_2) + 5\,(y_1 + y_4 + y_2 + y_0 + y_3):$$

which are severally equal to the sum of any vertical column, since

$$x_0 + x_1 + x_2 + x_3 + x_4 = 5x_2,$$

$$y_0 + y_1 + y_2 + y_3 + y_4 = 5y_2.$$

Hence we see that not only must *C* be the middle column, but the middle number of *C* must be $= 1 + x_2 + 5y_2$. From this it is easy to deduce that the number of different squares to be obtained from this method $= \left(\dfrac{4.3.2.1}{2}\right)^2$.

In a similar manner it may be shewn that the number of magic squares which can be formed of the numbers 1. 2. 3...9

$$= \left(\frac{2.1}{2}\right)^2 = 1.$$

The number which can be formed of the numbers 1. 2. 3 ... 49 by both methods

$$= 2.\left(\frac{7.6...3.2.1}{2}\right)^2 + \left(\frac{6.5...2.1}{2}\right)^2.$$

Let us now examine a little more particularly the nature of the method adopted in the above cases, and for this purpose let us revert to our original example. It will be seen that in that instance the column of x's in *B* is formed from that in *A* by rejecting the two first x's and throwing them to the bottom, their order being unchanged. The column of x's in *C* is formed from that in *B* in like manner, and so on for the rest.

In the second example the column of x's in *B* is formed from that in *A* by rejecting the first x only, and placing it at the bottom. The column of y's in the first case is formed by removing two of the y's from the foot of the preceding column and placing them in their order at the head of the new one, and similarly in the second case.

If we try the effect of rejecting the *three* first x's in the column, we shall obtain the same succession of columns as in the first case, but in the reverse order: and if we reject the

four first x's we shall have the same succession of columns as in the second example, likewise in the reverse order.

When the magic square is to contain 49 places, we may reject the first x in the generating column, and so obtain $\left(\frac{6.5...3.2.1}{2}\right)^2$ different squares. We may next reject the two first x's in the generating column, and so get $\left(\frac{7.6...2.1}{2}\right)^2$ additional squares; and lastly we may reject the three first x's, and so obtain $\left(\frac{7.6.5...2.1}{2}\right)^2$ squares.

But when the magic square contains 81 places, though we can make use of the two first of the above methods, yet when we reject the three first x's from the top of the column, and proceed by that rule, the method totally fails. It will be found that this failure is owing to the fact of the number of x's rejected being a divisor of the number of places in the side of the square. In this case however, if we reject the *four* first x's the method will succeed with certain limitations, *i.e.* provided the sum of the 2nd, 5th, and 8th places of x in the middle or generating column is equal to 12, and the sum of the like places of y is equal to the same number. Hence the number of squares obtained by this last method will be

$$= 4.\left(\frac{3.2.1 \times 6.5...2.1}{2}\right)^2,$$

and the whole number of magic squares of 81 places

$$= \left(\frac{9.8...2.1}{2}\right)^2 + \left(\frac{8.7...2.1}{2}\right)^2 + 4\left(\frac{3.2.1 \times 6.5...2.1}{2}\right)^2.$$

Generally if $2n + 1$ be a prime number, the whole number of squares of $(2n + 1)^2$ places which can be performed by the above methods

$$= (n - 1)\left\{\frac{(2n + 1)...3.2.1}{2}\right\}^2 + \left(\frac{2n...3.2.1}{2}\right)^2.$$

When the number of places in a side of the magic square is not a prime number, the rule for finding the number of squares to be obtained by the above methods is one of very great complexity; and, as the subject is of no practical importance, I shall content myself with merely indicating the method by which it is to be obtained. Let mr be the number of places in the side of a square where r is a prime number. If we reject nr of the x's from the top of the generating column, and the same number of y's from the

bottom, the method totally fails (n denoting any integer). If we reject *one* x from the top and one y from the bottom, the method fails partially, *i.e.* we shall obtain only

$$\left\{\frac{(nr-1).(nr-2)....2.1}{2}\right\}^2 \text{ squares.}$$

If we reject $(1+nr)\,x$'s and a like number of y's from the top and bottom respectively, there will again be a partial failure. If p denote the number of ways in which the number mr^2 can be made up by taking the sum of m of the numbers 0. 1. 2. ...$(mr-1)$, the number of squares to be deduced by rejecting $(1+nr)$ x's and y's respectively will be

$$=\left\{\frac{p\times(mr-m)(mr-m-1)....3.2.1}{2}\right\}.$$

If we reject a number of x's and y's respectively, greater than unity and not divisible by either nr or $nr-1$, the resulting number of squares will be

$$=\left(\frac{mr.(mr-1)....3.2.1}{2}\right)^2.$$

I propose to consider the case of magic squares of an even number of places, in a subsequent paper.

III.—ON THE THEORY OF DEVELOPMENTS. PART I.

By George Boole.

In a paper published in the *Philosophical Transactions* for the year 1844, Part II., I had occasion to investigate the expansion of the binomial $f(\pi+\rho)$, on the assumption that π and ρ are symbols operating on a certain subject u, and combining according to the law

$$\rho f(\pi)\,u = f\{\phi(\pi)\}\,\rho u \text{ .... (1).}$$

The result possesses a theoretic interest, because it shews the general form of the development of which Taylor's is a particular case; while at the same time it is, I conceive, of fundamental importance in the theory of differential equations and of equations of finite differences. For this application I must, however, refer to the paper above mentioned. My design here is to notice certain other deductions from the theorem in question, and to shew that the method by which it was obtained is generally applicable.

It may be proper, by way of introduction, to state in what sense such an expression as $f(\pi)$ is to be understood, when π is not a symbol of quantity.

In the first place, if $f(\pi) = \pi^m$, it is evident, that by $\pi^m u$ we are to understand the result obtained by operating with π upon u, then with π upon the result, and so on till the operation denoted by the symbol π shall have been m times performed.

If $f(\pi)$ is of the forms to which Maclaurin's theorem is applicable, it is evident that we must, in interpreting $f(\pi)$, suppose the expansion to be effected. Thus

$$(\sin \pi)\, u = \left(\pi - \frac{\pi^3}{1.2.3} + \frac{\pi^5}{1.2.3.4.5} - \&c.\right) u.$$

The legitimacy of the expansion of $f(\pi)$ is apparently independent of the nature of the symbol π; for as π operates solely on u, it may be regarded as commutative with respect to the constants in $f(\pi)$. We shall at any rate, in what follows, regard π as a symbol of this nature.

When $f(\pi)$ is not of the forms to which Maclaurin's expansion applies, it does not appear to be generally possible to interpret $f(\pi)\, u$. The laws of combination and of interpretation (when possible) to which $f(\pi)$ must be considered subject, have perhaps no other foundation than analogy.

In considering a binomial $f(\pi + \rho)$, we may, on writing η for $\pi + \rho$, expand in ascending powers of η, and then substitute $\pi + \rho$ for η in the result; thus

$$\begin{aligned}\sin(\pi + \rho)\, u &= \left(\eta - \frac{\eta^3}{1.2.3} + \&c.\right) u \\ &= \left(\pi + \rho - \frac{(\pi + \rho)^3}{1.2.3} + \&c.\right) u,\end{aligned}$$

but neither by Maclaurin's nor by Taylor's theorem can we obtain an expansion in ascending powers of ρ, unless π and ρ are commutative. The case in which π and ρ combine, according to the law (1), has been already referred to; but to render this analysis more complete, it may be proper to quote the investigation here.

Prop. Let π and ρ be distributive symbols which combine in subjection to the law

$$\rho f(\pi)\, u = \lambda f(\pi)\, \rho u \;\ldots\ldots\ldots\; (2),$$

λ being a functional symbol operating on π in such manner that $\lambda f(\pi) = f\{\phi(\pi)\}$, it is required to expand $f(\pi + \rho)$ in ascending powers of ρ.

We have
$$\left.\begin{aligned}\rho f(\pi)\, u &= \lambda f(\pi)\, \rho u \\ \rho^2 f(\pi)\, u &= \lambda^2 f(\pi)\, \rho^2 u \\ &\;\;\vdots \\ \rho^m f(\pi)\, u &= \lambda^m f(\pi)\, \rho^m u\end{aligned}\right\} \ldots\ldots\; (3).$$

Let $\pi+\rho=\eta$; then $f(\pi+\rho)\,u=f(\eta)\,u$. Now, as η operates solely on u, it is commutative with respect to the constants in $f(\eta)$; wherefore

$$\eta f(\eta)\,u=f(\eta)\,\eta u.$$

Or, dropping the subject u, and writing $\pi+\rho$ for η,

$$(\pi+\rho)f(\pi+\rho)=f(\pi+\rho)(\pi+\rho)\ldots\ldots(4).$$

Let $f(\pi+\rho)\,u=\Sigma f_m(\pi)\,\rho^m u$; then, still supposing u understood,

$$\begin{aligned}(\pi+\rho)f(\pi+\rho)&=\pi\Sigma f_m(\pi)\,\rho^m+\rho\Sigma f_m(\pi)\,\rho^m\\&=\Sigma\pi f_m(\pi)\,\rho^m+\Sigma\lambda f_m(\pi)\,\rho^{m+1}\ \text{ by (3).}\end{aligned}$$

Under the first Σ in the second member the coefficient of ρ^m is $\pi f_m(\pi)$, and under the second Σ the coefficient of ρ^m is $\lambda f_{m-1}(\pi)$; wherefore the aggregate coefficient of ρ^m is

$$\pi f_m(\pi)+\lambda f_{m-1}(\pi)\ldots\ldots\ldots\ldots(5).$$

Again, we have

$$\begin{aligned}f(\pi+\rho)(\pi+\rho)&=\Sigma f_m(\pi)\,\rho^m\pi+\Sigma f_m(\pi)\,\rho^{m+1}\\&=\Sigma f_m(\pi)\,\lambda^m\pi\rho^m+\Sigma f_m(\pi)\,\rho^{m+1},\end{aligned}$$

in which the aggregate coefficient of ρ^m is

$$f_m(\pi)\,\lambda^m\,\pi+f_{m-1}(\pi)\ldots\ldots\ldots\ldots(6).$$

Equating this with (5), we have

$$f_m(\pi)\,\lambda^m\pi+f_{m-1}(\pi)=\pi f_m(\pi)+\lambda f_{m-1}(\pi),$$

hence
$$f_m(\pi)=\frac{\lambda f_{m-1}(\pi)-f_{m-1}(\pi)}{\lambda^m\pi-\pi};$$

or separating the symbols of operation,

$$f_m(\pi)=\frac{(\lambda-1)f_{m-1}(\pi)}{(\lambda^m-1)\,\pi}\ldots\ldots\ldots(7),$$

which expresses the law of formation of the coefficients.

The first term, $f_0(\pi)$, is equal to $f(\pi)$. For let k be a symbol operating on π, in such manner that $kf(\pi)=f_0(\pi)$; then the first term of the expansion of $(\pi+\rho)f(\pi+\rho)$ is $k\pi f(\pi)$: but by (5) this term is $\pi f_0(\pi)=\pi kf(\pi)$; wherefore

$$k\pi f(\pi)=\pi kf(\pi),$$

wherefore π and k are *commutative.* It is hence evident that k can only operate as a constant multiplier, the value of which is independent of the form of $f(\pi)$. Let $f(\pi)=\pi$; then, since $f(\pi+\rho)=\pi+\rho$, it is evident that $k=1$; wherefore

$$f_0(\pi)=f(\pi)$$

in all cases, and the expansion is completely determined.

COR. If the symbols π and ρ combine, according to the law,

$$\rho f(\pi)\, u = f(\pi + \Delta\pi)\, \rho u,$$

$\Delta\pi$ being any constant increment; then

$$f(\pi + \rho) = f(\pi) + \frac{\Delta}{\Delta\pi} f(\pi)\, \rho + \frac{\Delta^2}{\Delta\pi^2} f(\pi)\, \frac{\rho^2}{1.2} + \&c. \ldots (8),$$

the interpretation of $\dfrac{\Delta}{\Delta\pi}$ being

$$\frac{\Delta}{\Delta\pi} f(\pi) = \frac{f(\pi + \Delta\pi) - f(\pi)}{\Delta\pi} \ldots\ldots (9),$$

for $\lambda f(\pi) = f(\pi + \Delta\pi)$. Hence $\lambda^m \pi = \pi + m\Delta\pi$, and (7) gives

$$f_m(\pi) = \frac{f_{m-1}(\pi + \Delta\pi) - f_{m-1}(\pi)}{m\Delta\pi}$$

$$= \frac{1}{m}\, \frac{\Delta}{\Delta\pi} f_{m-1}(\pi);$$

whence the theorem is manifest.

If $\Delta\pi$ vanishes, the symbols π and ρ are commutative, $\dfrac{\Delta}{\Delta\pi}$ becomes $\dfrac{d}{d\pi}$, and (8) is reduced to Taylor's theorem.

I proceed now to notice two remarkable deductions from the above theorem, each including several particular results of great interest.

Let us consider the expression $f\left\{x + \phi'\left(\dfrac{d}{dx}\right)\right\} u$, in which $\phi'\left(\dfrac{d}{dx}\right)$ denotes a function of the symbol $\dfrac{d}{dx}$, derived from a certain other arbitrary function $\phi\left(\dfrac{d}{dx}\right)$, in like manner as $\phi'(t) = \dfrac{d}{dt}\phi(t)$.

Now $\phi'\left(\dfrac{d}{dx}\right)$ is the limit to which $\phi'\left(\theta + \dfrac{d}{dx}\right)$ approaches as θ approximates to 0. Hence $f\left\{x + \phi'\left(\dfrac{d}{dx}\right)\right\} u$ is the limit of $f\left\{x + \phi'\left(\theta + \dfrac{d}{dx}\right)\right\} u$. But

$$f\left\{x + \phi'\left(\theta + \frac{d}{dx}\right)\right\} u = f\left\{x + \varepsilon^{\frac{d}{dx}\frac{d}{d\theta}}\phi'(\theta)\right\} u$$
$$= f(x + \rho)\, u,$$

$$\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}}\phi'(\theta) = \rho.$$

Now, since

$$\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}}\phi'(\theta)f(x)u = \varepsilon^{\frac{d}{d\theta}\frac{d}{dx}}f(x)\phi'(\theta)u$$
$$= f\left(x + \frac{d}{d\theta}\right)\varepsilon^{\frac{d}{d\theta}\frac{d}{dx}}\phi'(\theta)u,$$

we have, on substitution of ρ,

$$\rho f(x)u = f\left(x + \frac{d}{d\theta}\right)\rho u.$$

Hence, if in (8) we write x in the place of π, and suppose $\Delta x = \dfrac{d}{d\theta}$, we have

$$f(x+\rho)u = \left\{f(x) + \frac{\Delta}{\Delta x}f(x)\rho + \frac{1}{1.2}\frac{\Delta^2}{\Delta x^2}f(x)\rho^2 + \&c.\right\}u$$
$$= \varepsilon^{\frac{\Delta'}{\Delta x}\rho''}f(x)u \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

in which the symbol $\dfrac{\Delta'}{\Delta x}$ is accented to indicate that it refers to $f(x)$ only, and ρ'' is doubly accented to shew that it refers to u only.

Now, since $\Delta x = \dfrac{d}{d\theta}$, we have by (9),

$$\frac{\Delta'}{\Delta x} = \frac{\varepsilon^{\frac{d}{d\theta}\frac{d'}{dx}} - 1}{\dfrac{d}{d\theta}} = (\varepsilon^{\frac{d}{d\theta}\frac{d'}{dx}} - 1)\left(\frac{d}{d\theta}\right)^{-1};$$

moreover $\qquad \rho'' = \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\phi'(\theta).$

Substituting these values in (10), we find

$$f(x+\rho)u = \varepsilon^{(\varepsilon^{\frac{d}{d\theta}\frac{d'}{dx}} - 1)\left(\frac{d}{d\theta}\right)^{-1}\varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\phi'(\theta)}f(x)u$$
$$= \varepsilon^{\left\{\varepsilon^{\left(\frac{d'}{dx}+\frac{d''}{dx}\right)\frac{d}{d\theta}} - \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}}\right\}\left(\frac{d}{d\theta}\right)^{-1}\phi'(\theta)}f(x)u$$
$$= \varepsilon^{(\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}} - \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}})\phi(\theta)}f(x)u \ldots\ldots (11).$$

Since $\dfrac{d'}{dx} + \dfrac{d''}{dx} = \dfrac{d}{dx}$, to be taken with reference to both $f(x)$ and u; and since $\dfrac{d^{-1}}{d\theta}\phi'(\theta) = \phi(\theta)$. But

$$(\varepsilon^{\frac{d}{dx}\frac{d}{d\theta}} - \varepsilon^{\frac{d''}{dx}\frac{d}{d\theta}})\phi(\theta) = \phi\left(\theta + \frac{d}{dx}\right) - \phi\left(\theta + \frac{d''}{dx}\right);$$

whence $$f(x+\rho)\,u = \epsilon^{\phi\left(\theta+\frac{d}{dx}\right)-\phi\left(\theta+\frac{d''}{dx}\right)} f(x)\,u.$$

Let θ vanish, and putting for ρ its limiting value $\phi'\left(\frac{d}{dx}\right)$, we have
$$f\left\{x+\phi'\left(\frac{d}{dx}\right)\right\} u = \epsilon^{\phi\left(\frac{d}{dx}\right)-\phi\left(\frac{d''}{dx}\right)} f(x)\,u.$$

As, however, $\epsilon^{-\phi\left(\frac{d''}{dx}\right)}$ does not affect $f(x)$, we may transpose it, and remove the double accent; whence
$$f\left\{x+\phi'\left(\frac{d}{dx}\right)\right\} u = \epsilon^{\phi\left(\frac{d}{dx}\right)} f(x)\,\epsilon^{-\phi\left(\frac{d}{dx}\right)} u \;\ldots\ldots (12),$$
which is the first of the results in question.

Considering, secondly, the expression $f\left\{\frac{d}{dx}+\phi'(x)\right\}u$, if we write this in the form
$$f\left\{\frac{d}{dx}+\phi'(\theta+x)\right\}u = f\left\{\frac{d}{dx}+\epsilon^{x\frac{d}{d\theta}}\phi'(\theta)\right\}u,$$
and proceed as above, we finally get
$$f\left\{\frac{d}{dx}+\phi'(x)\right\}u = \epsilon^{-\phi(x)} f\left(\frac{d}{dx}\right)\epsilon^{\phi(x)}u \;\ldots\ldots (13),$$
which is the second of the theorems in question. Perhaps this result might be obtained more simply by induction. We shall now notice a few applications.

In the particular case in which $f\left\{\frac{d}{dx}+\phi'(x)\right\}$ is of the form $\left\{\frac{d}{dx}+\phi'(x)\right\}^{-1}$, we have, by (13),
$$\left\{\frac{d}{dx}+\phi'(x)\right\}^{-1}u = \epsilon^{-\phi(x)}\left(\frac{d}{dx}\right)^{-1}\epsilon^{\phi(x)}u \;\ldots\ldots (14),$$
which is the known solution of the linear differential equation of the first order.

If, in (12), $\phi'\left(\frac{d}{dx}\right)$ be of the form $\frac{d^{-1}}{dx}$, the expansion will stop at the second term, whatever may be the form of the function denoted by f.

For
$$\begin{aligned} f\left\{x+\left(\frac{d}{dx}\right)^{-1}\right\}u &= \epsilon^{\log\frac{d}{dx}} f(x)\,\epsilon^{-\log\frac{d}{dx}}u \\ &= \frac{d}{dx}f(x)\left(\frac{d}{dx}\right)^{-1}u \\ &= f'(x)\,u + f(x)\int u\,dx \;\ldots\ldots (15). \end{aligned}$$

In like manner the expansion of $f\left(\dfrac{d}{dx}+\dfrac{1}{x}\right)u$ stops at the second term; thus

$$f\left(\frac{d}{dx}+\frac{1}{x}\right)u=\varepsilon^{-\log x}f\left(\frac{d}{dx}\right)\varepsilon^{\log x}u$$
$$=\frac{1}{x}f\left(\frac{d}{dx}\right)xu$$
$$=f\left(\frac{d}{dx}\right)u+\frac{1}{x}f'\left(\frac{d}{dx}\right)u \;\ldots\ldots (16).$$

In (12) let $\phi'\left(\dfrac{d}{dx}\right)=\dfrac{d}{dx}$, we have

$$f\left(x+\frac{d}{dx}\right)u=\varepsilon^{\frac{1}{2}\frac{d^2}{dx^2}}f(x)\,\varepsilon^{-\frac{1}{2}\frac{d^2}{dx^2}}u$$
$$=\varepsilon^{\frac{1}{2}\left(\frac{d'}{dx}+\frac{d''}{dx}\right)^2-\frac{1}{2}\frac{d''^2}{dx^2}}f(x)\,u,$$

in which $\dfrac{d'}{dx}$ refers to $f(x)$, and $\dfrac{d''}{dx}$ to u. Hence

$$f\left(x+\frac{d}{dx}\right)u=\varepsilon^{\frac{1}{2}\frac{d'^2}{dx^2}+\frac{d'}{dx}\frac{d''}{dx}}f(x)\,u$$
$$=\varepsilon^{\frac{1}{2}\frac{d'^2}{dx^2}}\left\{f(x)+f'(x)\frac{d}{dx}+\frac{1}{1.2}f''(x)\frac{d^2}{dx^2}+\&c.\right\}u.$$

Let $\varepsilon^{\frac{1}{2}\frac{d^2}{dx^2}}f(x)=f_0(x)$; then

$$f\left(x+\frac{d}{dx}\right)u=\{f_0(x)+f_0'(x)\frac{d}{dx}+\frac{1}{1.2}f_0''(x)\frac{d^2}{dx^2}+\&c.\}\,u\ldots(17).$$

The coefficients of the expansion, after the first term, follow the law of Taylor's theorem, which is a remarkable circumstance, seeing that the symbols x and $\dfrac{d}{dx}$ are not commutative.

In (13) let $\phi'(x)=x$, we have

$$f\left(\frac{d}{dx}+x\right)u=\varepsilon^{-\frac{1}{2}x^2}f\left(\frac{d}{dx}\right)\varepsilon^{\frac{1}{2}x^2}u.$$

From this equation, after a troublesome reduction, I find

$$f\left(\frac{d}{dx}+x\right)u=\left\{f_0\left(\frac{d}{dx}\right)+f_0'\left(\frac{d}{dx}\right)x+\frac{1}{1.2}f_0''\left(\frac{d}{dx}\right)x^2+\&c.\right\}u \;\ldots\ldots (18),$$

in which $f_0\left(\dfrac{d}{dx}\right)$ is formed from $f\left(\dfrac{d}{dx}\right)$ in the same way

as $f_0(x)$ is formed from $f(x)$ in the preceding expansion. The developments are thus seen to be of precisely the same form, which again is a remarkable circumstance.

Some of the above deductions may be applied to the solution of differential equations. Thus, if we have an equation of the form

$$f_0(x)\,u + f_0'(x)\,\frac{du}{dx} + \frac{1}{1.2}f_0''(x)\,\frac{du^2}{dx^2} + \&c. = 0 \ldots (19),$$

in which $f_0(x)$ is a rational and integral function of x, we may at once place it under the form

$$f\left(x + \frac{d}{dx}\right)u = 0,$$

the form of the function denoted by f being determined by the relation

$$f(x) = \varepsilon^{-\frac{1}{2}\frac{d^2}{dx^2}} f_0(x) \quad \ldots\ldots\ldots (20).$$

Let $x + \dfrac{d}{dx} = \lambda$; then, supposing $a_1, a_2 \ldots\ldots$ to be the roots of the equation $f(x) = 0$, we have a system of equations of the form

$$(\lambda - a_1)\,u = 0 \quad \text{or} \quad (x - a_1)\,u + \frac{du}{dx} = 0,$$

$$(\lambda - a_2)\,u = 0 \quad \text{or} \quad (x - a_2)\,u + \frac{du}{dx} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

If $u_1 = 0$, $u_2 = 0, \ldots\ldots$ are the particular integrals thus obtained, then

$$u = c_1 u_1 + c_2 u_2 + \&c.$$

will be the complete integral.

Ex. Given $(x^2 - 5x + 7)\,u + (2x - 5)\,\dfrac{du}{dx} + \dfrac{d^2u}{dx^2} = 0.$

Here $f_0(x) = x^2 - 5x + 7$; whence

$$f(x) = \varepsilon^{-\frac{1}{2}\frac{d^2}{dx^2}} f_0(x) = \left(1 - \frac{1}{2}\,\frac{d^2}{dx^2} + \&c.\right) f_0(x)$$

$$= x^2 - 5x + 6$$

$$= (x-2)(x-3);$$

we have therefore the system

$$(\lambda - 2)\,u = 0, \quad \text{or} \quad (x - 2)\,u + \frac{du}{dx} = 0.$$

$$(\lambda - 3)\,u = 0, \quad \text{or} \quad (x - 3)\,u + \frac{du}{dx} = 0,$$

therefore $$u = c_1\,\varepsilon^{2x - \frac{x^2}{2}} + c_2\,\varepsilon^{3x - \frac{x^2}{2}}.$$

By (16) we may in like manner integrate any equation of the form

$$xf\left(\frac{d}{dx}\right)u + f'\left(\frac{d}{dx}\right)u = 0 \;\ldots\ldots\ldots(21).$$

There are some other equations, particularly in finite differences, which the above theorems enable us to solve. It is only however in connexion with the *symbolical* form of the linear equation, as discussed in the paper above referred to, that such applications can be reduced to a uniform and general system.

It might have been thought simpler, in the preceding investigations, to apply directly the principle by which the expansion of $f(\pi + \rho)$ was obtained, and thus to deduce the expansions of $f\left(x + \frac{d}{dx}\right)$ from the fundamental equation

$$\left(x + \frac{d}{dx}\right) f\left(x + \frac{d}{dx}\right) u = f\left(x + \frac{d}{dx}\right)\left(x + \frac{d}{dx}\right) u.$$

This method would indeed at once have given the law of derivation of the coefficients *after the first,* but it would still have been necessary, in order to obtain the first term, to adopt a process of reasoning similar to that which we have in reality employed. In illustration of this remark, let us take the expression $f\left(u + v\,\frac{d}{dx}\right) U$, u and v being functions of x, and let us seek to expand this in the form $\Sigma f_m(x)\,\frac{d^m}{dx}\,U$. We have

$$\left(u + v\,\frac{d}{dx}\right) f\left(u + v\,\frac{d}{dx}\right) U = f\left(u + v\,\frac{d}{dx}\right)\left(u + v\,\frac{d}{dx}\right) U,$$

$$\text{or } \left(u + v\,\frac{d}{dx}\right)\Sigma f_m(x)\left(\frac{d}{dx}\right)^m U = \Sigma f_m(x)\left(\frac{d}{dx}\right)^m\left(u + v\,\frac{d}{dx}\right) U \ldots(22).$$

Now the first member becomes

$$\Sigma\left\{uf_m(x)\left(\frac{d}{dx}\right)^m + vf_m'(x)\left(\frac{d}{dx}\right)^m + vf_m(x)\left(\frac{d}{dx}\right)^{m+1}\right\} U,$$

in which the aggregate coefficient of $\left(\frac{d}{dx}\right)^m$ is

$$uf_m(x) + vf_m'(x) + vf_{m-1}(x) \ldots\ldots\ldots(23).$$

Similarly we find, as the coefficient of $\left(\frac{d}{dx}\right)^m$ in the second member, the series

$$f_{m-1}(x) + f_m(x)(u + mv') + (m+1) f_{m+1}(x)\left(u' + \frac{m}{1.2} v''\right)$$
$$+ (m+2) f_{m+2}(x)\left\{u'' + \frac{m(m+1)}{1.2.3} v'''\right\} + \&c.$$

And, equating these results,

$$vf'_m(x) - mv'f_m(x) = (m+1) f_{m+1}(x)\left(u' + \frac{m}{1.2} v''\right)$$
$$+ (m+2) f_{m+2}(x)\left(u'' + \frac{m(m+1)}{1.2.3} v'''\right) + \&c. \ldots (24),$$

which is the law connecting the coefficients of the expansion.

In the case of $f\left(x + \frac{d}{dx}\right)$, we have $u = x$, $v = 1$, $u' = 1$, $v' = 0$, $u'' = 0$, &c.; whence, substituting in (24), we have

$$f'_m(x) = (m+1) f_{m+1}(x),$$
$$\text{or } f_{m+1}(x) = \frac{1}{m+1} f_m'(x).$$

This agrees with our previous result, but it leaves $f_0(x)$ undetermined.

In the case of $f\left(x \frac{d}{dx}\right) U$, we have $u = 0$, $v = x$, $v' = 1$, whence

$$xf_m'(x) - mf_m(x) = 0,$$

and, solving the equation,

$$f_m(x) = cx^m.$$

This shews that the expansion of $f\left(x \frac{d}{dx}\right) U$ is of the form $\Sigma a_m x^m \left(\frac{d}{dx}\right)^m U$, as is perhaps known.

In a subsequent part, it is intended to apply the method of this paper to the development of polynomials.

Lincoln, Jan. 3, 1845.

IV.—DEMONSTRATION OF A FUNDAMENTAL PROPOSITION IN THE MECHANICAL THEORY OF ELECTRICITY.

By William Thomson, B.A. St. Peter's College.

If a material point be in a position of equilibrium when under the influence of any number of masses attracting it or repelling it with forces which are inversely proportional to the square of the distance, the equilibrium will be unstable.*

* This theorem was first given by Mr. Earnshaw, in his Memoir on Molecular Forces, read at the Cambridge Philosophical Society, March 18, 1839. See vol. VII. of the *Transactions.*

The first thing to be proved is, that if the material point receive a slight displacement, there will in general be a moving force called into action.

Let O be the position of equilibrium; P any adjacent point; V the potential of the influencing masses, μ, at P, which point we suppose not to be contained within any portion of μ; U the value of V at O. Now it is shown by Gauss, in his Mémoire on General Theorems in Attraction, that V cannot have the constant value U through any finite volume, however small, adjacent to O, without having it for every point external to μ. But this is impossible, as may be shown in the following manner.

Let σ be a closed surface containing within it a quantity of matter, μ_1, consisting of any number of detached portions of μ, or of the whole of μ, if μ be a continuous mass. Let $d\sigma$ be an element of σ, and P the force due to the total action of μ, resolved in a direction perpendicular to $d\sigma$, which may be considered positive when directed towards the space within σ. Then, by a theorem demonstrated in this *Journal* (vol. III. p. 203), we have

$$\iint P d\sigma = 4\pi\mu,$$

the integrations being extended over the whole of σ. Hence P cannot be $= 0$ for every point of the surface σ, and therefore V cannot be constant for all the space exterior to μ.

Hence V cannot have the constant value U for every point of any finite volume, however small, adjacent to O.

Now let a sphere S be described round O as centre, with any radius a, sufficiently small that no portion of μ shall be included, and let P be any point of the surface S, and ds an element of the surface at P.

In the equations (3) and (4) of the article already referred to (vol. III. p. 202), let the sphere S be the surface there considered; let $v = V$, and $v_1 = \frac{1}{r}$, if $OP = r$.

Hence $P_1 = \frac{1}{a^2}$, $v_1 = \frac{1}{a}$, at every point of S;

$$m = \mu, \quad m_1 = 1, \quad \iiint v\,dm_1 = U.$$

Also $$\iint v_1 P ds = \frac{1}{a}\iint P ds = 0,$$

and $\iiint v_1 dm = 0$, since S does not contain any of the matter μ. We have therefore, by comparing (3) and (4),

$$0 = 4\pi U - \frac{1}{a^2}\iint V ds.$$

Therefore $$\iint V ds = 4\pi a^2 U,$$
which shows that the mean value for the surface of a sphere, of the potential of any external masses, is equal to the value at the centre. Let $$V = U + u.$$
Therefore $$\iint u ds = 0.$$

Now, as has already been shown, u cannot be $= 0$ for every point P adjacent to O, and therefore if the sphere pass through a point P' where u is negative, there must also be a point P'' in the surface, for which u is positive. But if we assume the potential of an attracting particle to be positive, the direction of the resultant force, resolved along any straight line, will be that in which V increases. Hence there will be a force towards O, for points displaced along OP', and from O, for points displaced along OP''. Hence if M, the material point in equilibrium at O, be displaced along OP'', the moving force generated will tend to remove it further from O, which is therefore an unstable position.

As an application of this theorem, let us consider the case of any number of material points repelling one another according to the inverse square of the distance, and contained in the interior of a rigid closed envelope. Let the system be in equilibrium when acted upon by attracting or repelling masses distributed in any manner without the envelope.

It will generally be possible that there may be a position or positions of equilibrium, in which at least some of the particles are not in contact with the surface. If now we suppose all the particles fixed except one, not in contact with the surface, the equilibrium of this particle is, as has been shown, unstable. Hence, generally, the equilibrium of the system is unstable if any of the particles be not in contact with the surface, and therefore in nature the particles cannot remain in such a position. There must, however, be some stable position or positions in which the particles can rest, but in such, all the particles must be in contact with the surface of the envelope. The sole condition of equilibrium in this case will be that the resultant force on each particle shall be in the direction of a normal to the surface, and directed towards the exterior space. If the number of particles be infinite, and there be one position in which the whole surface is covered, there can be no other in which this is the case, as is shown in the paper in this *Journal* already quoted (vol. III. p. 205); and it is also readily seen that this position will be stable, and that no other in which the surface is not entirely covered can be stable. In this case the particles

will be distributed according to the law of the intensity of electricity on the surface, the space within being conducting matter, and the masses without being any electrified bodies. If a mechanical theory be adopted, *electricity* will actually be a number of material points without weight, which repel one another according to the inverse square of the distance. Thus the result we have arrived at is, that there can be permanently no free electricity in the interior of a conducting body under any circumstances whatever.

If, as may happen through the influence of the exterior masses, there cannot be a position of equilibrium of the particles covering the whole surface, there will be a permanent distribution, in which part of the surface is uncovered. This however is never the case with electricity, as a certain quantity of latent electricity is then decomposed, so that the whole surface is covered with electricity, either positive or negative. All the above reasoning would still apply, if we considered the masses of some points to be negative, and of some positive, and the force between any two to be a repulsion equal to the product of their masses divided by the square of their distance.

Since every particle is on the surface, the whole *medium* (if it can be properly so called), will be an indefinitely thin stratum, the thickness being in fact the ultimate breadth of an atom or material point. If we suppose these atoms to be merely centres of force, the thickness will therefore be absolutely nothing, and thus the *fluid* will be absolutely compressible and inelastic. Any thickness which the stratum can have must depend on a force of elasticity, or on a force generated by the contact of material points, and in either case will therefore require an ultimate law of repulsion more intense than that of the inverse square,* when the distance is very small, and we therefore conclude that this cannot be the ultimate law of repulsion in any elastic fluid. As, however, all experiments yet made serve to confirm the fact that there is no electricity in the interior of conducting bodies, or that the stratum has absolutely no thickness, we conclude that there is no elasticity in the assumed electric fluid, and thus the law of force deduced independently by direct experiments, is confirmed.

* This agrees with a result of Mr. Earnshaw.

St. Peter's College, Jan. 16, 1845.

VI.—ON THE REDUCTION OF THE GENERAL EQUATION OF SURFACES OF THE SECOND ORDER.

By William Thomson, B.A., St. Peter's College.

In the following paper, by a simple assumption with reference to the coefficients in the general equation of a surface of the second order, the cubic, by means of which the three principal axes are determined is made to assume a very simple form, which enables us to prove the reality of the roots, and to find the limits between which they lie, with great ease. It also leads to a very simple analytical proof, that the three principal axes are at right angles to one another.

Let
$$Ax^2 + By^2 + Cz^2 + 2A'yz + 2B'zx + 2C'xy \\ + 2A''x + 2B''y + 2C''z = H,$$

or, for brevity, $H_2(x, y, z) + H_1(x, y, z) = H$,

be the equation to a surface of the second order.

Let
$$\left.\begin{array}{lll} A' = (gh)^{\frac{1}{2}}, & B' = (hf)^{\frac{1}{2}}, & C' = (fg)^{\frac{1}{2}}, \\ A = f + \alpha, & B = g + \beta, & C = h + \gamma, \end{array}\right\} \dots . (1),$$

from which we deduce

$$\left.\begin{array}{lll} f = \dfrac{B'C'}{A'}, & g = \dfrac{C'A'}{B'}, & h = \dfrac{A'B'}{C'}, \\ \alpha = A - \dfrac{B'C'}{A'}, & \beta = B - \dfrac{C'A'}{B'}, & \gamma = C - \dfrac{A'B'}{C'}, \end{array}\right\} \dots (2);$$

which express real determinate values for f, g, &c. in terms of the given coefficients. Making the substitutions (1), we find
$$H_2 = \alpha x^2 + \beta y^2 + \gamma z^2 + (f^{\frac{1}{2}}x + g^{\frac{1}{2}}y + h^{\frac{1}{2}}z)^2 \dots . (a).$$

Now we may define a principal axis to be a line such that, if it be taken for the axis of x', and the axes of y' and z' be in the plane perpendicular to it, the products $x'y'$ and $x'z'$ shall disappear in the transformed expression H_2. This will be ensured if the product $x'y'$ vanishes for every point in the plane $x'y'$, and for every position of this plane passing through the principal axis OX'; a definition equivalent to the one in which a principal axis is defined as an axis which is perpendicular to its diametral plane.

Let l, m, n be the direction-cosines of a principal axis OX'; l', m', n' those of any line OY' perpendicular to it; x', y' the co-ordinates of any point in the plane $X'OY'$;

x, y, z the co-ordinates of the same point referred to the original axes: we have

$$x = lx' + l'y',$$
$$y = mx' + m'y',$$
$$z = nx' + n'y'.$$

Therefore $\quad \alpha x^2 + \beta y^2 + \gamma z^2 + (f^{\frac{1}{2}}x + g^{\frac{1}{2}}y + h^{\frac{1}{2}}z)^2$

$$= (\alpha l^2 + \beta m^2 + \gamma n^2)\, x'^2 + (\alpha l'^2 + \beta m'^2 + \gamma n'^2)\, y'^2$$
$$+ 2\,(\alpha ll' + \beta mm' + \gamma nn')\, x'y'$$
$$+ \{(f^{\frac{1}{2}}l + g^{\frac{1}{2}}m + h^{\frac{1}{2}}n)\, x' + (f^{\frac{1}{2}}l' + g^{\frac{1}{2}}m' + h^{\frac{1}{2}}n')\, y'\}^2;$$

which becomes $\quad Px'^2 + P'y'^2,$

if we put for brevity

$$S = f^{\frac{1}{2}}l + g^{\frac{1}{2}}m + h^{\frac{1}{2}}n \;\ldots\ldots\ldots\; (3),$$
$$P = S^2 + \alpha l^2 + \beta m^2 + \gamma n^2 \;\ldots\ldots\; (4),$$

and similarly for $l'm'n'$, and if we assume the coefficient of $x'y' = 0$, which gives

$$(Sf^{\frac{1}{2}} + \alpha l)\, l' + (Sg^{\frac{1}{2}} + \beta m)\, m' + (Sh^{\frac{1}{2}} + \gamma n)\, n' = 0.$$

If OX' be a principal axis, this must hold for all values of l', m', n' consistent with

$$ll' + mm' + nn' = 0;$$

we must therefore have

$$\frac{Sf^{\frac{1}{2}} + \alpha l}{l} = \frac{Sg^{\frac{1}{2}} + \beta m}{m} = \frac{Sh^{\frac{1}{2}} + \gamma n}{n}:$$

therefore each member is

$$= l\,(Sf^{\frac{1}{2}} + \alpha l) + m\,(Sf^{\frac{1}{2}} + \beta m) + n\,(Sf^{\frac{1}{2}} + \gamma n),$$
$$= S^2 + \alpha l^2 + \beta m^2 + \gamma n^2,$$
$$= P.$$

Therefore

$$\left.\begin{aligned} l &= \frac{Sf^{\frac{1}{2}}}{P - \alpha}, \\ m &= \frac{Sg^{\frac{1}{2}}}{P - \beta}, \\ n &= \frac{Sh^{\frac{1}{2}}}{P - \gamma}, \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (5).$$

Hence, by (3),

$$S\left(\frac{f}{P - \alpha} + \frac{g}{P - \beta} + \frac{h}{P - \gamma} - 1\right) = 0;$$

and therefore, unless $S = 0$, which, on account of (5), would

require $P = a = \beta = \gamma$, a case that will be considered below, we must have

$$\frac{f}{P-a} + \frac{g}{P-\beta} + \frac{h}{P-\gamma} - 1 = 0 \;\ldots\ldots (6),$$

which determines P. This equation, being a cubic, gives three values for P. Now, from equations (2) it follows that f, g, h must either be all positive or all negative; the former being the case when two or more of A', B', C', and the latter when one or three are negative.* Hence, if a, β, γ be in descending order of magnitude, and e be an indefinitely small quantity, and if we substitute

$$a - e, \quad \beta + e, \quad \text{and} \quad \beta - e, \quad \gamma + e,$$

for P in the first member of (6), the first and second values will have contrary signs, and so will the third and fourth. Hence the cubic has a real root between a, β, and another between β, γ, and its remaining root must therefore also be real, and between ∞ and a, or between γ and $-\infty$. The first is obviously the case when f, g, and h are positive, and the second when they are negative.

Let P_1, P_2 be any two roots of (6), and let $l_1, m_1, n_1, l_2, m_2, n_2$ be the corresponding values of l, m, n deduced from equations (5). Writing down (6) for each value, and subtracting, we have

$$(P_1 - P_2)\left\{\frac{f}{(P_1-a)(P_2-a)} + \frac{g}{(P_2-\beta)(P_2-\beta)} + \frac{h}{(P_1-\gamma)(P_2-\gamma)}\right\} = 0.$$

If P_1 be different from P_2, the second factor must vanish, or, by (5),

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \;\ldots\ldots\ldots (b).$$

Hence any two of the axes determined by equations (6) and (5) are at right angles, and therefore the three must form a rectangular system. If we take it for axes of co-ordinates, and if P, Q, R be the three roots of (6), the equation to the surface becomes

$$Px^2 + Qy^2 + Rz^2 + H_1 = H,$$

accents being omitted. If none of the quantities P, Q, R vanishes, we may obviously, by changing the origin, make H_1 disappear, and the equation will be reduced to the form

$$Px^2 + Qy^2 + Rz^2 = H',$$

which is the equation of surfaces of the second order referred to principal axes through the centre.

* Hence (6) is a particular case of a certain equation of any order, which has been shown by M. M. Plana and Liouville to have all its roots real. See Moigno, *Calc. Int.* p. 296.

If in this equation H' be $= 0$, the surface represented will be either a point or a cone, according as P, Q, and R have the same or different signs. Excluding this case, we may, without losing generality, consider H' as positive.

The equation will then represent an ellipsoid if P, Q, R be all positive; a hyperboloid of one sheet if one of them only be negative, and a hyperboloid of two sheets if two of them be negative. If all three be negative, the surface will be imaginary.

Hence, from equation (6) we infer that if $f, g, h, \alpha, \beta, \gamma$, be all positive the surface will be an ellipsoid, and if they be all negative it will be imaginary.

In addition to these we have the following cases.

I. $\dfrac{f}{-\alpha} + \dfrac{g}{-\beta} + \dfrac{h}{-\gamma} - 1 > 0.$

(1) $fgh > 0$.

$\alpha > 0, \quad \beta > 0, \quad \gamma < 0,$ Ellipsoid,
$\alpha > 0, \quad \beta < 0, \quad \gamma < 0,$ Hyperboloid of one sheet,
$\alpha < 0, \quad \beta < 0, \quad \gamma < 0,$ Hyperboloid of two sheets.

(2) $fgh < 0$.

$\alpha > 0, \quad \beta > 0, \quad \gamma > 0,$ Hyperboloid of one sheet,
$\alpha > 0, \quad \beta > 0, \quad \gamma < 0,$ Hyperboloid of two sheets,
$\alpha > 0, \quad \beta < 0, \quad \gamma < 0,$ Imaginary.

II. $\dfrac{f}{-\alpha} + \dfrac{g}{-\beta} + \dfrac{h}{-\gamma} - 1 < 0.$

(1) $fgh > 0$.

$\alpha > 0, \quad \beta > 0, \quad \gamma < 0,$ Hyperboloid of one sheet,
$\alpha > 0, \quad \beta < 0, \quad \gamma < 0,$ Hyperboloid of two sheets.
$\alpha < 0, \quad \beta < 0, \quad \gamma < 0,$ Imaginary.

(2) $fgh < 0$.

$\alpha > 0, \quad \beta > 0, \quad \gamma > 0,$ Ellipsoid,
$\alpha > 0, \quad \beta > 0, \quad \gamma < 0,$ Hyperboloid of one sheet,
$\alpha > 0, \quad \beta < 0, \quad \gamma < 0,$ Hyperboloid of two sheets.

These tests enable us, when A, B, C, A', B', C' are given numerically, to find the nature of the surface represented, provided it has a centre, by calculating $\alpha, \beta, \gamma, f, g, h$ from equations (2).

If the surface have not a centre, it can belong to neither of the four cases considered above, and we must therefore have

$$\frac{f}{\alpha}+\frac{g}{\beta}+\frac{h}{\gamma}+1=0,$$

which is the condition that one root of (6) may be $=0$. If we substitute for α, β, γ, their values, by (1), and clear of fractions, this becomes

$$ABC+2fgh-Agh-Bhf-Cfg=0,$$

or $$ABC+2A'B'C'-AA'^2-BB'^2-CC'^2=0,$$

which agrees with the condition given in Gregory's *Solid Geometry*, p. 64.

This is the condition that must be satisfied in the case in which it is impossible to make H_1 vanish, from the general equation, by any finite change in the position of the origin, as may be readily verified.

If the surface be of revolution, two of the roots of (6) must be equal. Hence each of the two must be equal to one of the quantities α, β, γ, on account of the limits found above for the roots. Hence, clearing of fractions, we find for the conditions

$$\alpha=\beta=\gamma,$$

and the remaining root will be given by

$$f+g+h-(P-\alpha)=0.$$

Hence, restoring the original constants, we have, in the case of surfaces of revolution,

$$A-\frac{B'C'}{A'}=B-\frac{C'A'}{B'}=C-\frac{A'B'}{C'}=Q=R,$$

which agrees with the condition given by *Gregory*, p. 109,

and $$P=Q+\frac{B'C'}{A'}+\frac{C'A'}{B'}+\frac{A'B'}{C'},$$

or $$=A+B+C-2Q.$$

The formulæ which have been proved above furnish us with a very simple proof of the following theorem of Chasles.

The principal axes of a cone touching a surface of the second order, are perpendicular to the three confocal surfaces of the second order which intersect in the vertex.

Let $$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1,$$

be the equation to the surface. That of the tangent cone through ξ, η, ζ is

$$\left(\frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}-1\right)\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1\right)=\left(\frac{\xi x}{a}+\frac{\eta y}{b}+\frac{\zeta z}{c}-1\right)^2.$$

Here $$H_2=\left(\frac{\xi x}{a}+\frac{\eta y}{b}+\frac{\zeta z}{c}\right)^2-K\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right),$$

where $$K=\frac{\xi^2}{a}+\frac{\eta^2}{b}+\frac{\zeta^2}{c}-1.$$

Hence, comparing with (a), we have

$$f=\frac{\xi^2}{a^2},\quad g=\frac{\eta^2}{b^2},\quad h=\frac{\zeta^2}{c^2},$$

$$\alpha=-\frac{K}{a},\quad \beta=-\frac{K}{b},\quad \gamma=-\frac{K}{c}.$$

Therefore (6) becomes

$$\frac{\xi^2}{a(aP+K)}+\frac{\eta^2}{b(bP+K)}+\frac{\zeta^2}{c(cP+K)}=1;$$

but $$\frac{\xi^2}{aK}+\frac{\eta^2}{bK}+\frac{\zeta^2}{cK}=1+\frac{1}{K}.$$

Hence, by subtraction,

$$\frac{\xi^2}{a+\dfrac{K}{P}}+\frac{\eta^2}{b+\dfrac{K}{P}}+\frac{\zeta^2}{c+\dfrac{K}{P}}=1 \quad \ldots\ldots\ldots (c).$$

Also, equations (5) give

$$\frac{l\left(a+\dfrac{K}{P}\right)}{\xi}=\frac{m\left(b+\dfrac{K}{P}\right)}{\eta}=\frac{n\left(c+\dfrac{K}{P}\right)}{\zeta}.$$

Hence l, m, n, are the direction-cosines of a normal at ξ, η, ζ, to the surface represented by (c) when either of the three values of $\dfrac{K}{P}$ is used in that equation. If the value $> a$ be used, the surface represented will be the ellipsoid confocal with the given surface which passes through the vertex of the cone; the value between a and b corresponds to the confocal hyperboloid of one sheet; and the value between b and c to the confocal hyperboloid of two sheets. Thus we infer that these three surfaces intersect at right angles in the

vertex of the cone, and that the three principal axes of the cone touch their lines of intersection.

The same theorem might be proved separately for the different cases of surfaces without centres; but, as these are only extreme cases of central surfaces, we may infer the truth of the theorem for them, as whatever is true in general, is true in limiting cases, provided the result remain definite.

St. Peter's College, Jan. 11, 1845.

V.—ON CERTAIN INTEGRAL TRANSFORMATIONS.

By B. Bronwin.

This paper is a continuation of one bearing the same title printed in the 12th number of this *Journal*, and the references are made to the transformations there given when the number is below (21).

$$\text{Let } y = \frac{(1+b)\, x\sqrt{(1-x^2)}}{\sqrt{(1-c^2x^2)}}, \quad b = \sqrt{(1-c^2)}, \quad k = \frac{1-b}{1+b},$$

$$\text{or } c = \frac{2k^{\frac{1}{2}}}{1+k} \; \dots (f).$$

$$\text{We find } \sqrt{(1-y^2)} = \frac{1-(1+b)\,x^2}{\sqrt{(1-c^2x^2)}}, \quad \sqrt{(1-k^2y^2)} = \frac{1-(1-b)\,x^2}{\sqrt{(1-c^2x^2)}},$$

$$\sqrt{(1-y^2)}\,.\,\sqrt{(1-k^2y^2)} = \frac{1-2x^2+c^2x^4}{1-c^2x^2}, \quad \frac{\sqrt{(1-y^2)}}{\sqrt{(1-k^2y^2)}} = \frac{1-(1+b)\,x^2}{1-(1-b)\,x^2},$$

$$\sqrt{(1-c^2x^2)} = \frac{k\sqrt{(1-y^2)}+\sqrt{(1-k^2y^2)}}{1+k},$$

$$2x^2 = 1 + ky^2 - \sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)};$$

$$\frac{dx}{\sqrt{(1-x^2)}\,\sqrt{(1-c^2x^2)}} = \frac{1+k}{2}\,\frac{dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}}.$$

Multiplying the last by any combination of those which precede it (and the resulting form will shew the combination employed), we easily obtain

$$\frac{x^{\frac{1}{2}}dx}{(1-x^2)^{\frac{1}{4}}\,(1-c^2x^2)^{\frac{3}{4}}} = \left(\frac{1+k}{2}\right)^{\frac{3}{2}} \frac{y^{\frac{1}{2}}dy}{\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \; \dots (21).$$

$$\frac{dx}{x^{\frac{1}{2}}\,(1-x^2)^{\frac{3}{4}}\,(1-c^2x^2)^{\frac{1}{4}}} = \left(\frac{1+k}{2}\right)^{\frac{1}{2}} \frac{dy}{y^{\frac{1}{2}}\sqrt{(1-y^2)}\,\sqrt{(1-k^2y^2)}} \; \dots (22).$$

These are E, F, by (1) and (2), respectively.

$$\frac{x^{\frac{3}{2}}dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{4}}} = \frac{1}{2}\left(\frac{1+k}{2}\right)^{\frac{1}{2}}\left\{\frac{(1+ky^2)\,dy}{y^{\frac{1}{2}}\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} - \frac{dy}{y^{\frac{1}{2}}}\right\}\ldots(23).$$

$$\frac{x^{\frac{5}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{3}{4}}} = \frac{1}{2}\left(\frac{1+k}{2}\right)^{\frac{3}{2}}\left\{\frac{(1+ky^2)\,y^{\frac{1}{2}}dy}{\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}} - y^{\frac{1}{2}}dy\right\}\ldots(24).$$

It is easily seen from (1), (2), that the two last are E, F; but it must be understood that they may contain circular or algebraic functions.

$$\frac{\sqrt{\{1-(1+b)x^2\}}}{\sqrt{\{1-(1-b)x^2\}}}\,\frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}} = \frac{1+k}{2}\,\frac{dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{3}{4}}}\ldots(25).$$

$$\frac{\sqrt{\{1-(1-b)x^2\}}}{\sqrt{\{1-(1+b)x^2\}}}\,\frac{dx}{\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}} = \frac{1+k}{2}\,\frac{dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ldots(26).$$

which are E, F, by (5) and (6).

$$\frac{x^{\frac{1}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}} = \frac{1}{2}\left(\frac{1+k}{2}\right)^{\frac{1}{2}}\left\{\frac{ky^{\frac{1}{2}}dy}{\sqrt{(1-k^2y^2)}} + \frac{y^{\frac{1}{2}}dy}{\sqrt{(1-y^2)}}\right\}\ldots(27).$$

We find

$$\frac{1}{\sqrt{(1-c^2x^2)}} = \frac{\sqrt{(1-k^2y^2)}-k\sqrt{(1-y^2)}}{1-k}, \quad \frac{1}{2x^2} = \frac{1+ky^2+\sqrt{(1-y^2)}\sqrt{(1-k^2y^2)}}{(1+k)^2y^2}.$$

And, by the aid of these,

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}} = \frac{(1+k)^{\frac{1}{2}}}{2^{\frac{1}{2}}(1-k)}\left\{\frac{dy}{y^{\frac{1}{2}}\sqrt{(1-y^2)}} - \frac{kdy}{y^{\frac{1}{2}}\sqrt{(1-k^2y^2)}}\right\}\ldots(28).$$

The two last are obviously E, F. And from these we easily derive the four following by means of the values of $2x^2$ and $\frac{1}{2x^2}$; but the second members, which are obviously E, F, are too complex to put down :

$$\left.\begin{array}{cc} \dfrac{x^{\frac{5}{2}}dx}{(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}} & \dfrac{x^{\frac{3}{2}}dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}} \\ \dfrac{dx}{x^{\frac{3}{2}}(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}} & \dfrac{dx}{x^{\frac{5}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}} \end{array}\right\}\ldots\ldots(29).$$

From the transformation (c) in the former paper, we easily find

$$\frac{(1-c)^{\frac{1}{2}}(1+c)^{\frac{3}{4}}dx}{(1-x)^{\frac{1}{4}}(1-cx)^{\frac{1}{4}}(1+x)^{\frac{1}{2}}(1+cx)} = \frac{2^{\frac{3}{4}}dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{4}}} \ldots\ldots(30).$$

$$\frac{(1+c)^{\frac{3}{4}}}{(1-c)^{\frac{1}{2}}}\frac{dx}{(1-x)^{\frac{3}{4}}(1-cx)^{\frac{3}{4}}(1+x)^{\frac{1}{2}}} = \frac{2^{\frac{3}{4}}dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{3}{4}}} \ldots(31).$$

These are E, F, by (7) and (8). And, from the same source,

$$\frac{(1+cx)\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}} = \left(\frac{1-c^2}{4}\right)^{\frac{1}{4}}\left(\frac{2}{1+c}\right)\frac{dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}} \ldots(32).$$

This is an E, F by (11). But if we make $1-c^2x^2=v^2$, we see that $\dfrac{x\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}}$ is an E, F; therefore

$$\frac{dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{1}{2}}} \text{ is an } E, F \ldots\ldots\ldots(33).$$

The transformation (6) gives

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{1}{4}}} = \frac{-(1-c^2)^{\frac{1}{4}}dy}{(1-y^2)^{\frac{3}{4}}(1-c^2y^2)^{\frac{1}{2}}} \ldots\ldots (34),$$

$$\frac{dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}(1-c^2x^2)^{\frac{3}{4}}} = \frac{-dy}{(1-c^2)^{\frac{1}{4}}(1-y^2)^{\frac{3}{4}}} \ldots\ldots\ldots(35).$$

To return now to the transformation (c),

$$\frac{\left(\frac{1-c}{2}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)dx}{(1+x)^{\frac{1}{4}}(1-x)\,(1-cx)^{\frac{1}{4}}(1+cx)} = \frac{y^{\frac{1}{2}}dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}} \ldots (36),$$

an E, F by (9);

$$\frac{\left(\frac{2}{1-c}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)dx}{(1+x)^{\frac{3}{4}}(1-x)^{\frac{1}{2}}(1-cx)^{\frac{3}{4}}} = \frac{dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{4}}} \ldots\ldots (37),$$

an E, F by (35).

$$\frac{(1-c)^{\frac{1}{4}}(1+c)^{\frac{1}{2}}dx}{(1-x)^{\frac{1}{2}}(1+x)^{\frac{3}{4}}(1-cx)^{\frac{1}{4}}(1+cx)^{\frac{1}{2}}} = \frac{2^{\frac{3}{4}}dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}} \ldots (38),$$

an E, F by (34).

$$\left(\frac{1-c^2}{4}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)\frac{dx}{(1-x^2)^{\frac{1}{4}}(1-cx)^{\frac{1}{4}}(1+cx)^{\frac{5}{4}}}=\frac{y^{\frac{1}{4}}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ldots(39),$$

an E, F by (27).

$$\frac{\left(\frac{2}{1-c}\right)^{\frac{1}{4}}\left(\frac{1+c}{2}\right)dx}{(1-x)^{\frac{3}{4}}(1+x)^{\frac{1}{2}}(1-cx)^{\frac{1}{2}}(1+cx)^{\frac{1}{4}}}=\frac{dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ldots(40),$$

an E, F by (33).

We now turn to the transformation (e) last paper, from which we derive

$$\frac{1+c}{\sqrt{(1-c)}}\,\frac{\sqrt{(1-cx^2)}\,dx}{x^{\frac{1}{2}}\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{-dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ldots.(41),$$

an E, F by (33).

$$\frac{\sqrt{(1-c^2)}\,\sqrt{(1+cx^2)}\,dx}{(1-cx^2)(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{-y^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}}\ldots.(42),$$

an E, F by (9).

$$\frac{\sqrt{(1-c^2)}\,\sqrt{(1+cx^2)}\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{-dy}{y^{\frac{1}{4}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{1}{4}}}\ldots.(43),$$

an E, F by (34).

$$\frac{(1+c)^{\frac{3}{2}}}{(1-c)^{\frac{1}{2}}}\,\frac{(1-cx^2)\,dx}{(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}(1+cx^2)^{\frac{1}{2}}}=\frac{-dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{1}{2}}(1-k^2y^2)^{\frac{3}{4}}}\ldots(44),$$

an E, F by (35).

$$\frac{(1-c)^{\frac{1}{2}}(1+c)\,x^{\frac{1}{2}}\,dx}{(1-cx^2)(1-x^2)^{\frac{1}{4}}(1-c^2x^2)^{\frac{1}{4}}}=\frac{-y^{\frac{1}{2}}\,dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ldots.(45),$$

an E, F by (10).

$$\frac{1+c}{\sqrt{(1-c)}}\,\frac{(1-cx^2)\,dx}{x^{\frac{1}{2}}(1-x^2)^{\frac{3}{4}}(1-c^2x^2)^{\frac{3}{4}}}=\frac{-dy}{y^{\frac{1}{2}}(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{2}}}\ldots(46),$$

an E, F by (11).

$$\frac{(1-c)(1+c)^{\frac{1}{2}}(1+cx^2)^{\frac{1}{2}}\,x^{\frac{1}{2}}\,dx}{(1-cx^2)\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{-dy}{(1-y^2)^{\frac{1}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ldots.(47),$$

an E, F by (7).

$$\frac{(1+c)^{\frac{1}{4}}}{1-c}\;\frac{(1-cx^2)\,dx}{x^{\frac{1}{2}}\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}\sqrt{(1+cx^2)}}=\frac{-dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{3}{4}}}\ldots(48),$$

an E, F by (8).

$$\frac{\sqrt{(1+c)}\sqrt{(1+cx^2)}\,dx}{x^{\frac{1}{2}}\sqrt{(1-x^2)}\sqrt{(1-c^2x^2)}}=\frac{-dy}{(1-y^2)^{\frac{3}{4}}(1-k^2y^2)^{\frac{1}{4}}}\ldots\ldots(49),$$

an E, F by (6).

VI.—ON CERTAIN CONTINUED FRACTIONS.

By PERCIVAL FROST, M.A., St. John's College.

To shew that

$$e^x=\frac{1}{1-}\,\frac{x}{1+}\,\frac{x}{2-}\,\frac{x}{3+}\,\frac{2x}{4-}\,\frac{2x}{5+}\cdots\cdots\frac{rx}{2r+1+}\,\frac{(r+1)x}{2r+2-}.$$

Let $$y=e^x=\frac{1}{1+y_1},$$

$$\frac{dy_1}{dx}=-\frac{d_xy}{y^2}=-1-y_1.$$

Let $y_1=a_1x$, $a_1=-1$, neglecting terms in x.

Assume $$y_1=-\frac{x}{1+y_2};$$

then $$\frac{dy_1}{dx}=-1-y_1=-\frac{1}{1+y_2}+\frac{\frac{dy_2}{dx}}{(1+y_2)^2},$$

$$x\frac{dy_2}{dx}=(1+y_2)\left\{1-(1+y_2)-y_1(1+y_2)\right\}$$
$$=(1+y_2)(x-y_2).$$

If $y_2=a_2x$, $a_2=1-a_2$, neglecting x.

Assume $$y_2=\frac{x}{2+y_3},$$

$$x\frac{dy_2}{dx}=-\frac{x}{2+y_3}-\frac{x^2\frac{dy_3}{dx}}{(2+y_3)^2};$$

then $(2+y_3+x)(2x+xy_3-x)=x(2+y_3)-x^2\dfrac{dy_3}{dx}$,

$$x\frac{dy_3}{dx}=-x(1+y_3)-y_3(2+y_3)\ldots\ldots\ldots(1);$$

$y_3=a_3x$ approximately

$a_3=-1-2a_3$, $a_3=-\frac{1}{3}$.

Let
$$y_3 = -\frac{x}{3 + y_4},$$

$$\frac{dy_3}{dx} = -\frac{1}{3 + y_4} + \frac{x\dfrac{dy_4}{dx}}{(3 + y_4)^2}$$

$$-(1 + y_3) - \frac{y_3}{x}(2 + y_3) = -\frac{1}{3 + y_4} + \frac{x\dfrac{dy_4}{dx}}{(3 + y_4)^2},$$

$$x\frac{dy_4}{dx} = -(3 + y_4)\{y_3(3 + y_4) + 2 + y_4\} + 2.(3 + y_4) - x$$
$$= -(3 + y_4)\,y_4 + (2 + y_4)\,x,$$
$$a_4 = -3a_4 + 2 \text{ as before, } a_4 = \tfrac{2}{4}.$$

Let
$$y_4 = \frac{2x}{4 + y_5};$$

whence, as before, $x\dfrac{dy_5}{dx} = -x(2 + y_5) - (4 + y_5)\,y_5$.

According to law observed in equations for y_5, y_3,

let
$$x\frac{dz}{dx} = -(r + z)\,x - (2r + z)\,z,$$
$$z = \beta x, \quad \beta = -r - 2r\beta.$$

Let
$$z = -\frac{rx}{2r + 1 + z_1},$$

$$x\frac{dz}{dx} = -\frac{rx}{2r + 1 + z_1} + \frac{rx^2\dfrac{dz_1}{dx}}{(2r + 1 + z_1)^2};$$

Thus
$$\frac{rx^2\dfrac{dz_1}{dx}}{(2r + 1 + z_1)^2} = -(r + z)\,x - (2r + 1 + z)\,z;$$

and
$$rx\frac{dz_1}{dx} = -\{r(2r + 1 + z_1) - rx\}(2r + 1 + z_1)$$
$$+ r\{(2r + 1)^2 + z_1(2r + 1) - rx\},$$

$$x\frac{dz_1}{dx} = -(2r + 1 + z_1)\,z_1 + x(r + 1 + z_1),$$
$$z_1 = \beta_1 x,$$
$$\beta_1 = -(2r + 1)\,\beta_1 + r + 1,$$
$$\beta_1 = \frac{r + 1}{2r + 2}.$$

Let
$$z_1 = \frac{(r + 1)\,x}{2r + 2 + z_2},$$

$$\frac{dz_1}{dx} = \frac{r+1}{(2r+2+z_2)} - \frac{(r+1)\,x\,\dfrac{dz_2}{dx}}{(2r+2+z_2)^2};$$

whence $$\frac{(r+1)\,x\,\dfrac{dz_2}{dx}}{2r+2+z_2} = r+1+(r+1)(2r+1+z_1),$$

$$-(r+1)(2r+2+z_2)-(r+1)\,x,$$

$$x\frac{dz_2}{dx} = (2r+2+z_2)(z_1-z_2-x)$$

$$= (r+1)\,x-(z_2+x)(2r+2+z_2)$$

$$= -(r+1+z_2)\,x-\{2(r+1)+z_2\}\,z_2,$$

the same form as for $x\dfrac{dz}{dx}$; therefore the law observed is true,

$$y_{2r} = \frac{rx}{2r+y_{2r+1}},\quad y_{2r+1} = -\frac{rx}{2r+1+y_{2r+2}},$$

and $$e^x = \frac{1}{1-}\,\frac{x}{1+}\,\frac{x}{2-}\,\frac{x}{3+}\,\frac{2x}{4-}\,\frac{2x}{5+}\,\frac{3x}{6-}\,\frac{3x}{7+}\cdots\cdots$$

In the same manner $l_\varepsilon(1+x)$ can be expressed in a continued fraction, $$y = l_\varepsilon(1+x),$$

$$\frac{dy}{dx} = \frac{1}{1+x},\quad y = \frac{x}{1+y_1},$$

$$x(1+x)\frac{dy_1}{dx} = (1+y_1)\,x-(1+y_1)\,y_1,$$

$$a_1 = \tfrac{1}{2},\quad y_1 = \frac{x}{2+y_2},$$

and $$x(1+x)\frac{dy_2}{dx} = x-(2+y_2)\,y_2,$$

$$y_2 = \frac{x}{3+y_3},\quad x(1+x)\frac{dy_3}{dx} = (4+y_3)\,x-(3+y_3)\,y_3.$$

The equation for y_{2r+1} is

$$x(1+x)\frac{dy_{2r-1}}{dx} = (r^2+y_{2r-1})\,x-y_{2r-1}(y_{2r-1}+2r-1),$$

for $$y_{2r}\quad x(1+x)\frac{dy_{2r}}{dx} = rx-ry_{2r}(2+y_{2r});$$

and $$y_{2r-1} = \frac{rx}{2+y_{2r}},\quad y_{2r} = \frac{rx}{2r+1+y_{2r+1}},$$

the result being that

$$l_\epsilon(4x) = \frac{x}{1+}\;\frac{x}{2+}\;\frac{x}{3+}\;\frac{2x}{2+}\;\frac{2x}{5+}\;\frac{3x}{2+}\;\frac{3x}{7+}\;\frac{4x}{2+}\;\dots\dots$$

The method applies with great facility to many other functions.

For $\tan^{-1} x = y$, $\quad y = \dfrac{x}{1+y_1}$,

$$x(1+x^2)\frac{dy_1}{dx} = (1+y_1)x^2 - (1+y_1)y_1;$$

and, proceeding as before,

$$x(1+x^2)\frac{dy_r}{dx} = (r^2+y_r)x^2 - (2r-1+y_r)y_r,$$

and

$$y_r = \frac{r^2x^2}{2r+1+y_{r+1}},$$

$$\tan^{-1} x = \frac{x}{1+}\;\frac{x^2}{3+}\;\frac{2^2x^2}{5+}\;\frac{3^2x^2}{7+}\;\frac{4^2x^2}{9+}\;\dots\dots$$

For $y = \tan x$, $\quad x\dfrac{dy_r}{dx} = -x^2 - (2r-1+y_r)y_r$

$$y_r = -\frac{x^2}{2r-1+y_{r+1}},$$

$$\tan x = \frac{x}{1-}\;\frac{x^2}{3-}\;\frac{x^2}{5-}\;\frac{x^2}{7-}\;\dots\dots$$

In the case of $(1+x)^n$, we obtain by the law of formation,

$$x(1+x)\frac{dy_{2r+1}}{dx} = (n-r+n\,y_{2r+1})x - r\,y_{2r+1}(2+y_{2r+1})x$$

whence

$$x(1+x)\frac{dy_{2r+2}}{dx} = \{(r+1)(n+r+1)+(n+1)\,y_{2r+2}\}x$$

and

$$-\,y_{2r+2}(2r+1+y_{2r+2}),$$

$$x(1+x)\frac{dy_{2r+3}}{dx} = -\{n-(r+1)+n\,y_{2r+3}\}x - (r+1)y_{2r+3}(2+y_{2r+3}),$$

which proves that the law holds generally.

Thus,

$$y_{2r} = \frac{(n+r)x}{2+y_{2r+1}},$$

$$y_{2r+1} = -\frac{(n-r)x}{2r+1+y_{2r+2}},$$

$$(1+x)^n = \frac{1}{1-}\;\frac{nx}{1+}\;\frac{(n+1)x}{2-}\;\frac{(n-1)x}{3+}\;\frac{(n+2)x}{2-}\;\frac{(n-2)x}{5+}\;\frac{(n+3)x}{2-},$$

which is the required development.